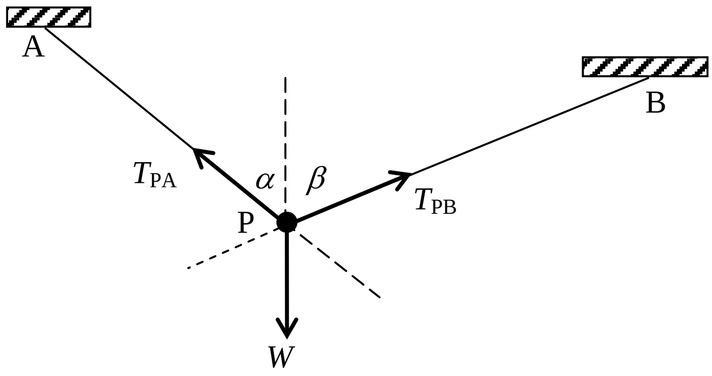
A transition from equilibrium to motion

A small object is held in static equilibrium by the use of two light inextensible strings. What happens to the tension in one string when the other is cut?



Two light strings PA and PB are fixed at A and at B and are attached to a particle of weight W at P. The system is in equilibrium.

We first find the tension in AP.

Resolving horizontally gives $T_{PA}\sin\alpha = T_{PB}\sin\beta$ (1)

Resolving vertically gives $T_{PA}\cos\alpha + T_{PB}\cos\beta = W$ (2)

Solving for $\beta \neq 0$ gives $T_{PA} = \dfrac{W\sin\beta}{\sin(\alpha+\beta)}$ (3)

[Alternatively, resolve in the direction perpendicular to PB giving $T_{PA}\cos(\alpha+\beta-90) - W\sin\beta = 0$. Also see below.]

Suppose that the string PB is cut. The system is no longer in equilibrium but, starting from rest, the object now swings in a circular arc centre A. This is the same situation as that of a pendulum at the top of its swing with the string taut.

The equation of motion in the radial direction is

$$W\cos\alpha - T_{\mathrm{PA}} = -\frac{mv^2}{\mathrm{AP}} \qquad (4)$$

where m is the mass of the object and v is its speed.

At the instant the string is cut, $v = 0$ so $T_{\mathrm{PA}} = W\cos\alpha$ (5)

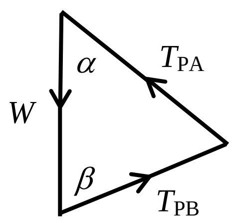
We can see that typically (1) and (5) have different values so there is an instantaneous change in the tension in the string AP.

When is there no change in tension?

We could equate (3) and (5) and solve but the result can be found more readily as follows. We know the tension in AP must be $W\cos\alpha$ after the string is cut so we need the same tension before that happens. All we need is a zero component in the direction of AP of the tension in PB, i.e. $\alpha + \beta = 90°$.
[Checking, if we substitute $\alpha + \beta = 90°$ in (3) we get

$$T_{\mathrm{PA}} = \frac{W\sin(90-\alpha)}{\sin 90} = W\cos\alpha$$ as required.]

It is worth noting that result (3) comes immediately from considering a triangle of forces, a method that requires much lower level manipulative skills with trigonometric identities.



Some notes on appropriate approximations when using calculus to find areas, volumes, moments of inertia and centres of mass.

1. Calculus methods often involve writing down an approximate expression involving a small quantity before taking the limit of a sum as this quantity tends to zero. The theory involved in deciding on the required accuracy of the approximation is not known to most A level students and so they can have some difficulty in deciding what accuracy to use and tend to try to remember particular cases instead of a general principle.

   I think it is of benefit to give them a principle to guide them, even though they cannot derive it. There is also a nice fallacy which, when they have seen it, can help them avoid a common error.

2. The rule they should try to apply is that their approximate expression should agree with the exact result 'to the 1st order of small quantities'. In other words the *discrepancy* should not involve δx, δy etc but only be in terms of $\delta x^2, \delta x^3$ or $\delta x \delta y$ etc. These higher order terms make no contribution when the limit of the sum is taken.

A few examples:

3. Suppose we want to find the mass of a circular disc, centre O and radius *a*, that has a surface density of *kr*, where *k* > 0 and *r* is the distance from O.

   The standard approach is first to find the area of an annulus of radius between *r* and $(r+\delta r)$, shown shaded.

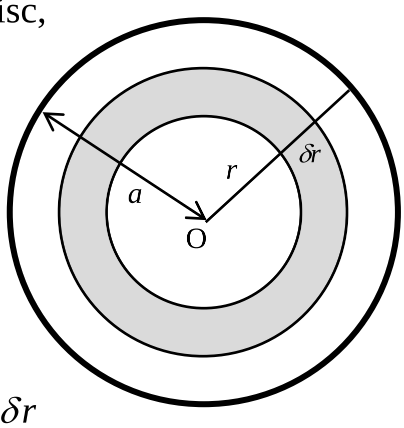
   

   Diagram 1

   The exact area of the shaded region is

   $\pi\left(r+\delta r\right)^2 - \pi r^2 = 2\pi r\delta r + \pi\delta r^2$ and this is $2\pi r\delta r$ to 1st order.

   The 'usual' approach is say that the annulus is approximately a rectangle of length the perimeter of the annulus, $2\pi r$, and width δr, and it can be seen that this method gives the result to 1st order.

   Note that there is no point in saying that a better approximation for the area of the annulus is that it is a rectangle of length $2\pi(r+\frac{\delta r}{2})$ and width δr as this gives an area of $2\pi r\delta r + \pi\delta r^2$ and we only need the 1st order term, $2\pi r\delta r$.

The required mass is approximately $\sum 2\pi r\delta r \times kr$ and is exactly

$$\lim_{\delta r \to 0} \sum \left(2\pi k r^2\right)\delta r = \int_0^a \left(2\pi k r^2\right) \mathrm{d}r \text{ etc.}$$

4. Diagram 2 shows a part of a general curve. In the diagram, δx and δy have their usual meanings, δs is the distance from A to B along the curve and δl the length of the line segment AB.

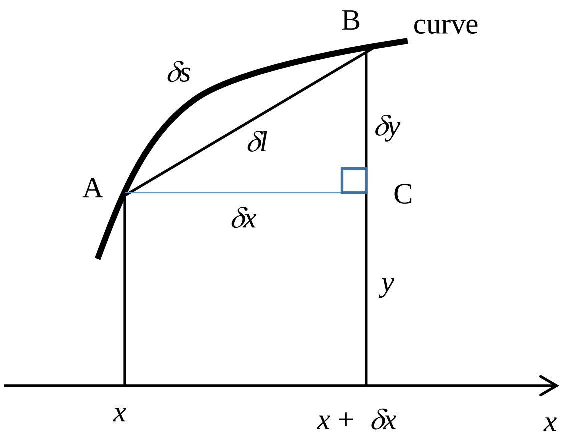


Diagram 2

First we note that as B moves down the curve towards A, $\delta x \to 0$ and $\lim_{\delta x \to 0} \dfrac{\delta s}{\delta l} = 1$. This means that δl and δs agree to 1st order and we can use the line segment AB instead of the arc AB in our calculations and still get a result correct to 1st order.

In passing, we can note that when we find the area under the curve by summing the areas of the elementary rectangles and take the area of each to be $y\delta x$, we are actually using a 1st order approximation. In this case, the area neglected in each elementary contribution is approximately $\dfrac{1}{2}\delta x \delta y$ and so makes no contribution when we take the limit of the sum.

5. Now let us look at thin slices of a sphere formed by slicing the sphere using parallel planes perpendicular to a diameter, as shown in Diagram 3.

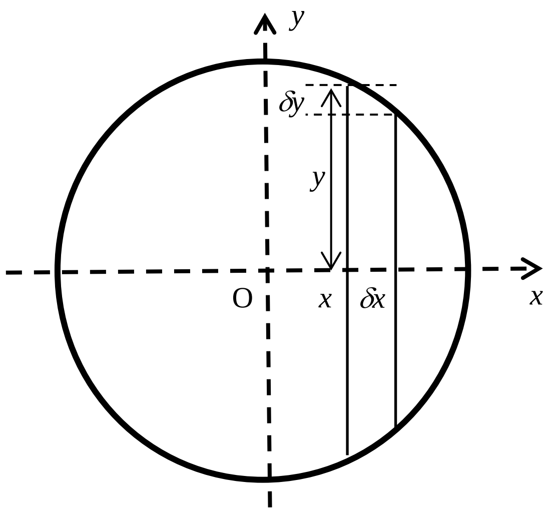


Diagram 3

If we want to know the volume, δV, of such a slice we can see

$$\pi\left(y - \delta y\right)^2 \delta x < \delta V < \pi y^2 \delta x \text{ so}$$

$$\pi y^2 \delta x - 2\pi y \delta y \delta x + \pi \delta y^2 \delta x < \delta V < \pi y^2 \delta x$$

and δV may be taken to be $\pi y^2 \delta x$ to 1st order. This is just the volume of a disc of radius y and thickness δx.

6. When dealing with arc lengths or surface areas we have to take account of the fact that δs (and hence δl) are not generally equal to δx to 1st order. This is clear from a study of Diagrams 2 and 4.

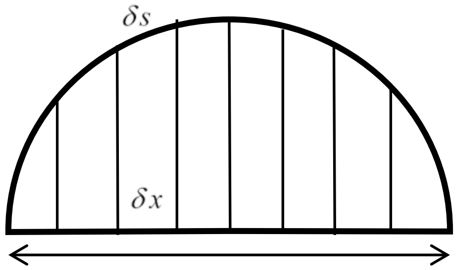


Diagram 4

Diagram 4 shows a semicircle of radius 1 divided into a number of vertical strips. The arc length of each strip is δs and the width of each strip is δx. Now it is clear that $\sum \delta s = \pi$ but $\sum \delta x = 2$ and that this argument is not affected in any way if we increase the number of strips and take the limit of the sum.

7. The surface area of a slice of a right circular cone is easily found. The curved surface area of a cone with slant length *l* and radius *r* is πrl. The surface area, δS, of the elementary section shown in Diagram 5 is exactly $\pi(r+\delta r)(l+\delta l)-\pi rl$ so $\delta S = \pi(r\delta l + l\delta r + \delta r\delta l)$.

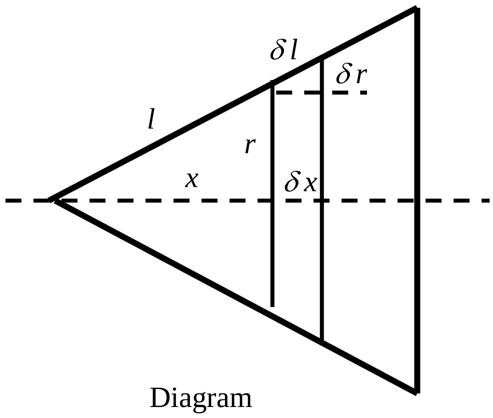


Diagram 5

Now, for a cone $\dfrac{\delta r}{\delta l} = \dfrac{r}{l}$ so $l\delta r = r\delta l$ and $\delta S = 2\pi r\delta l + \pi\delta r\delta l$.

This means that $\delta S = 2\pi r\delta l$ to 1st order.

This result can be used to find the surface area of general curves using the result above in 4. that δl can be used in place of δs as they will give expressions for areas that agree to 1st order.

Loose stone chips may damage your car?

20 mph speed limit

A road has recently been re-surfaced with tar and stone chips.

Investigate whether 20 mph is a realistic speed limit to avoid loose stone chips hitting the car behind.

Hints:

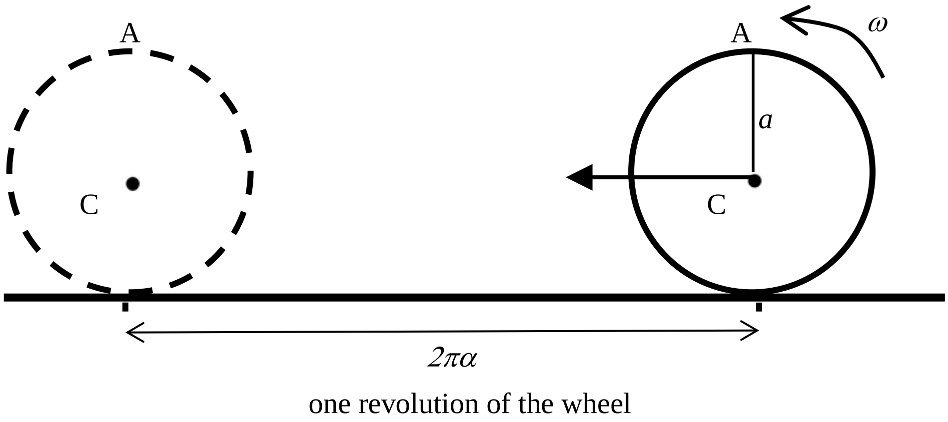
There are different ways in which these stone chips (now called stones) can be projected. These include coming off a revolving tyre or being ‘pinged’ like a tiddly-wink. I suggest you try to model only the motion of stones thrown directly from the rotating tyre in a vertical plane perpendicular to the rear axle..

You will find it useful to investigate the motion of a wheel rolling without slipping on a horizontal surface.

Taking the forward speed of the axle to be v m s^{-1}, the radius of the wheel with tyre to be a m and the angular speed of the wheel to be ω rad s^{-1}, find the relationship between v, a and ω and hence the velocity of any part of the tyre relative to the road.

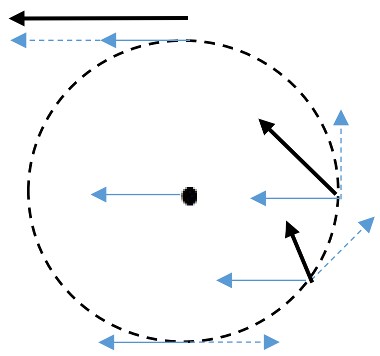
Notes for teachers

1. According to the nature of the group and the outcomes you are seeking, you may wish to give less or more help (e.g. a diagram) at this stage.
2. Some of the possible dangers such as a stone being 'pinged' like a tiddly-wink are very hard to model and probably best avoided, even though they may well present most danger. Similarly, it is probably better to concentrate only on the stones that go straight along the road.
   There may be a need to travel slowly because of the greater danger of slipping. This is not being investigated here.
3. The motion of a non-slipping, rolling wheel may not have been seen before by your students and it has several properties they will find interesting.



The time taken for 1 revolution of the wheel is $\frac{2\pi}{\omega}$ s and the distance travelled in this time is $2\pi a$ m, so the linear speed, v m s^{-1}, of the centre C is $2\pi a \div \frac{2\pi}{\omega} = a\omega$ so $v = a\omega$.

4. Of course, the linear speed of a point on the circumference of the tyre *relative to the axle* is also v but in a direction tangential to the tyre. The velocity *relative to the road* of any particle on the circumference of the tyre is the *sum of the velocity of the axle with the velocity of the particle relative to the axle* and this gives a number of interesting results which can be neatly illustrated on a diagram.



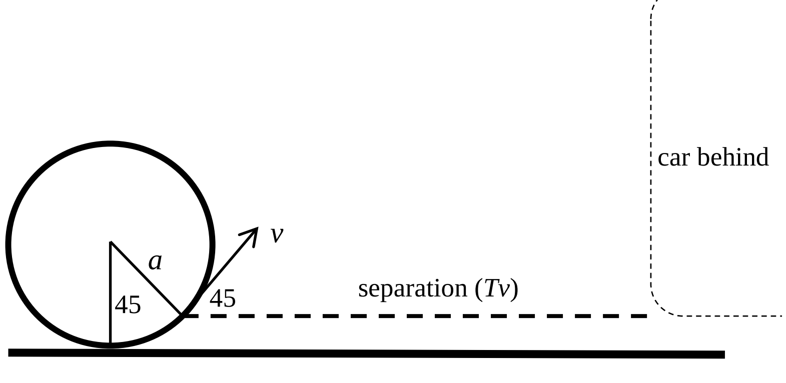
   It is particularly worth noting that
   a. The point of contact with the road is instantaneously at rest (thus accounting for a rolling wheel doing no work against friction)
   b. No point on the wheel is moving to the right relative to the road. It follows that any object thrown off tangential to the wheel cannot be travelling backwards relative to the road; this is not obvious when you look at stones or snow coming from the back wheels of a car. Of course, this argument does not apply to a slipping wheel.
5. It might be worth mentioning a flanged wheel such as on a railway truck on which points are travelling backwards relative to the rail. It might also be worth introducing the idea that at any instant every point on the wheel is travelling in a circle with centre the instantaneously stationary point of contact of the wheel with the road.
6. To answering the original question one has now to model the situation. In the work below, we shall assume that the wheel is not slipping. Since any point on the circumference of the tyre is travelling in a direction tangential to the tyre, it seems reasonable to assume that the stones would come off in this direction. At what positions of the wheel can stones come off without hitting the wheel arch or mudflap? How much distance is reasonable to allow between cars (say a 2 s gap)? One could also consider the relative danger of collisions at different heights.
7. My car has a (wheel + tyre) radius of 33 cm and stones leaving the tyre at angles greater than about $30°$ to the horizontal would hit the bodywork and be deflected down (but this angle is nearer $45°$ for some SUVs).

8. One simple way of modelling the situation is to say that the worst case is not as bad as assuming the stone leaves the tyre with no forward speed and has a vertical speed of *v*. Neglecting air resistance, if the stone is projected from a height *a* above the road it is in the air for time *T*, where *T* is the positive root of $-a = vT - \frac{1}{2}gT^2$.

   This gives $T = \dfrac{v + \sqrt{v^2 + 2ag}}{g}$. Assuming the following car is at least 2 s behind, means that we need $\dfrac{v + \sqrt{v^2 + 2ag}}{g} < 2$ for the stone to reach the ground before the next car reaches it. This gives $v < g - \frac{1}{2}a$ and, using the value of *a* as 0.33 for my car gives $v \approx 9.6$. [20 mph ≈ 9 m s^{-1}]. Note that taking *a* = 0 makes little difference.

9. Another simple (and more accurate) way to analyse the problem is to argue that if the cars are travelling at the same speed then one may disregard the forward speed of each and simply require that the range of the stone thrown from the front car *as if the axle were at rest* is less than the distance to the car behind. Neglecting air resistance, the maximum horizontal range is when the stone is projected at $45°$ to the horizontal and since a stone leaving a wheel of radius *a* at $45°$ to the horizontal is only about 0.3*a* above the ground at the start (under 10 cm for my car), it seems reasonable to use the *horizontal* range of the stone (especially as when the stone falls below a height of 0.3*a* it would pass under the car behind). We shall also assume that a stone projected at $45°$ to the horizontal is not deflected down by the rear of the car from which it comes.

   The following diagram indicates this situation.



All we need now is the maximum range, $\frac{v^2}{g}$, to be less than the separation distance when the time separation is T s.

This gives $\frac{v^2}{g} < Tv$ so $v < Tg$.

When $T = 2$, $v \approx 20$, which is about 45 mph. [Using an angle of projection of 30° gives $v \approx 23$ m s^{-1}, which is about 51 mph.]

10. The results in 9 might suggest that the speed restriction was not because of the perceived dangers from stones thrown off the tyres in the way we have considered. However, the critical speed is directly proportional to T so for lower time separations between the cars the critical speed reduces. With a 1 s separation the critical speed is about 23 mph.

The Moment of Inertia of a chocolate orange segment

1. It is sometimes possible to find the MoI of an object about a given axis by using arguments based on the additive nature of MoI and/or on the distribution of the mass of the object about the axis.
2. A simple example is that of the MoI of a segment of a thin uniform circular disc about an axis perpendicular to the disc through its centre.

   Diagram 1 (i) shows a uniform disc of radius *a* with a segment of angle 2α. The axis of rotation is through O perpendicular to the plane of the disc.

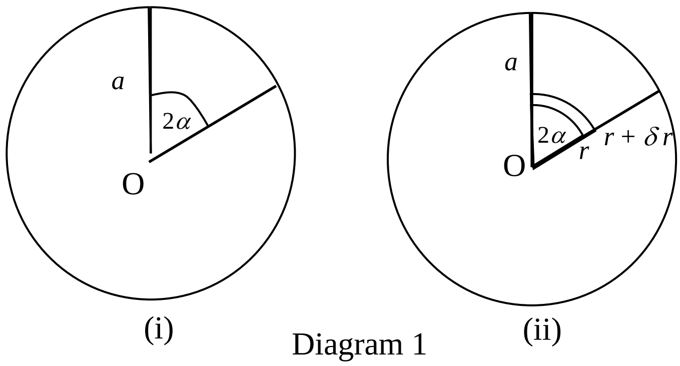
   

   Diagram 1

   Suppose the disc has mass *m*. It is easy to show that its MoI about the axis perpendicular to the disc through O is $\frac{1}{2}ma^2$.

   Since the disc is uniform and because every segment shares the same distribution of mass about the axis of rotation and because MoI is additive it must follow that the MoI of the segment about the same axis through O is $\frac{2\alpha}{2\pi} \times \frac{1}{2}ma^2 = \frac{1}{2}\left(\frac{\alpha m}{\pi}\right)a^2$.

   Now, the mass of the segment is $\frac{2\alpha}{2\pi}m = \frac{\alpha m}{\pi}$ so if we write the MoI of both whole disc and segment in terms of the mass, *M*, of the *object* they are both $\frac{1}{2}Ma^2$ (with $M = m$ for the whole disc and $M = \frac{\alpha m}{\pi}$ for the segment).

   For this example, the argument used corresponds to the working in the usual derivation using calculus, which is shown in 3.

3. Suppose the surface density of the material of the disc is σ.

   Consider an elementary annular region of width δr a distance *r* from O, as shown in Diagram 1 (ii).

The area of this region is approximately $2\alpha r\delta r$ and the mass of this region is approximately $2\alpha r\sigma\delta r$.

The MoI of this region about the axis is approximately $2\alpha r\sigma\delta r \times r^2 = 2\alpha\sigma r^3\delta r$.

The MoI of the whole segment is $\lim_{\delta r\to 0}\sum 2\alpha\sigma r^3\delta r = \int_0^a 2\alpha\sigma r^3 \mathrm{d}r = 2\alpha\sigma\frac{a^4}{4}$

$= \frac{1}{2}\left(\sigma\alpha a^2\right)\times a^2$. (*)

Now the mass, M, of the segment is $\frac{1}{2}a^2 \times 2\alpha \times \sigma = \sigma\alpha a^2$ so the MoI is $\frac{1}{2}Ma^2$.

It can easily be seen from (*) that the factor $\left(\sigma\alpha a^2\right)$ is just the mass of the segment (which is in direct proportion to α); writing this mass as M gives the same formal equation for any segment. One special case is when $\alpha = \pi$ and the segment becomes the whole disc.

4. A similar example is in the MoI of parts of a uniform square about an axis perpendicular to its plane and through its centre. Diagrams 2 (i) and (ii) show two ways in which the square can be divided into 4 parts.

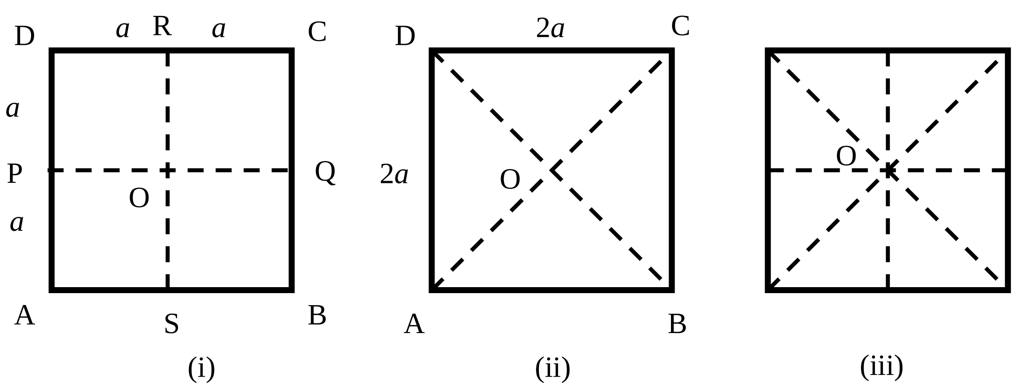


Diagram 2

The MoI of the whole square ABCD is easily found. Suppose ABCD has side $2a$ and mass M.

Referring to Diagram 2 (i) and using the result for the MoI of a uniform rod length 2*a* about its centre, the MoI of ABCD about PQ is $\frac{1}{3}Ma^2$. By symmetry, the MoI of ABCD about RS is also $\frac{1}{3}Ma^2$.

Using the perpendicular axis theorem, the MoI of ABCD about an axis through O perpendicular to the plane is $\frac{1}{3}Ma^2 + \frac{1}{3}Ma^2 = \frac{2}{3}Ma^2$.

Using a similar argument to that applied above to the disc, in both Diagrams 2 (i) and (ii) the MoI of the 4 parts are equal and each part has ¼ of the mass of the whole square so each has a MoI of $\frac{2}{3}Ma^2$, where M is the mass of a quarter part of the square.

5. A similar argument can be used for the parts of the square in Diagram 2 (iii); the MoI of each part about the axis through O perpendicular to the plane is again $\frac{2}{3}Ma^2$ but, this time, M is one eighth of the mass of the square.

6. There are, of course, limitations to the application of this method as the subdivisions must have the same mass distribution about the axis.

7. Arguments such as these are not only neat, they can also save a lot of work. It is quite easy to deal with the case in Diagram 2 (i) from 1st principles but the cases in (ii) and (iii) are not so easy. Case (i) can be argued as follows.

   Consider the square ORDP: this has side *a* and so, using the result above for the MoI of a square of side *2a* about an axis perpendicular to its plane and through its centre, we get a MoI of $\frac{2}{3}M\left(\frac{a}{2}\right)^2 = M\frac{a^2}{6}$, where M is the mass of ORDP. The centre of square ORDP is $\frac{a}{\sqrt{2}}$ from O so using the parallel axis theorem the MoI of ORDP about O is

   $M\frac{a^2}{6} + M\left(\frac{a}{\sqrt{2}}\right)^2 = Ma^2\left(\frac{1}{6}+\frac{1}{2}\right) = \frac{2}{3}Ma^2$ as required.

8. **The chocolate orange**

Let us now apply the method to the 3D chocolate orange given that the MoI of a uniform sphere radius *a* and mass *M* about a diameter is $\frac{2}{5}Ma^2$.

Suppose the whole sphere is made from uniform material.

If the mass of the sphere is *m* its MoI about a diameter is $\frac{2}{5}ma^2$. If the segment is $\frac{1}{n}$th of the sphere it has mass $\frac{m}{n}$. If we consider the MoI of the segment about an axis along its straight edge, using the additive nature of MoI and the fact that, relative to the axis of rotation, the segment has its mass distributed in the same way as the whole sphere, the segment must have a MoI of $\frac{1}{n}\times\frac{2}{5}ma^2$.

Since $\frac{1}{n}\times\frac{2}{5}ma^2 = \frac{2}{5}\times\frac{m}{n}a^2$, if we write the MoI of both whole sphere and segment in terms of the mass, *M*, of the object they are both $\frac{2}{5}Ma^2$, with

$M = m$ for the sphere and $M = \frac{m}{n}$ for the segment.

9. It is also true that the segments in any combination 'slid' along the axis of rotation will still have the same total MoI about that axis.

10. Arguing the result for the MoI of the segment from 1st principles takes quite a lot of work and an outline proof is given here.

Consider a segment of angle 2α taken from a sphere of radius *a* centre O. Diagram 3 shows the segment seen from the side and also a typical cross-section. The axis of rotation of the segment is along the line AOB.

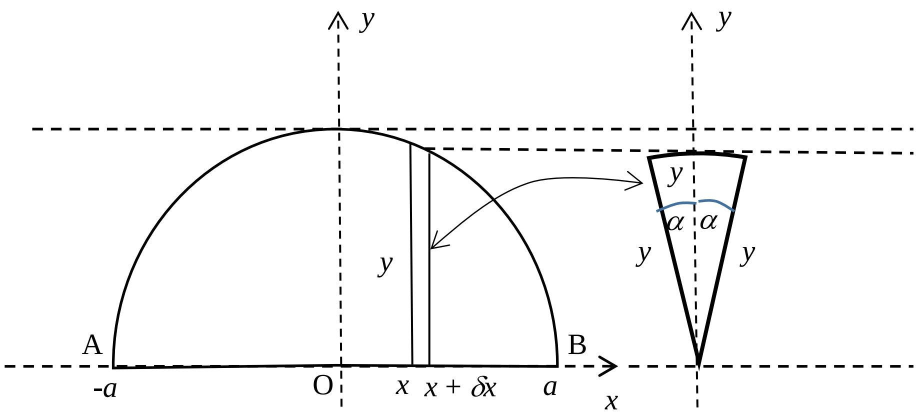


Diagram 3

Consider a thin section of the segment, distance *x* from O of height *y* and thickness δx, as shown in Diagram 3 and suppose the density of the material is ρ. We have $x^2 + y^2 = a^2$.

The mass of the thin section is approximately

$$\frac{1}{2} \times 2\alpha y^2 \times \delta x \times \rho = \alpha \rho y^2 \delta x.$$

Using the result derived above in 2 (and 3), the MoI of this thin section about AB is $\frac{1}{2}Ma^2$ which is approximately $\frac{1}{2}\left(\alpha\rho y^2 \delta x\right) \times y^2 = \frac{1}{2}\alpha\rho y^4 \delta x$.

We require $\lim\limits_{\delta x \to 0} \sum \frac{1}{2}\alpha\rho y^4 \delta x = \int_{-a}^{a} \left(\frac{1}{2}\alpha\rho y^4\right) \mathrm{d}x = 2\int_{0}^{a} \left(\frac{1}{2}\alpha\rho\left(a^2 - x^2\right)^2\right) \mathrm{d}x$

$$= \alpha\rho \int_0^a \left(a^4 - 2a^2x^2 + x^4\right) \mathrm{d}x = \alpha\rho\left[a^4x - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5\right]_0^a$$

$$= \alpha\rho a^5\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{8}{15}\alpha\rho a^5 = \frac{2}{5} \times \left(\frac{4}{3}\alpha\rho a^3\right)a^2$$

Now the mass of the segment *M*, is $\frac{2\alpha}{2\pi} \times \frac{4}{3}\pi a^3 \rho = \frac{4}{3}\alpha\rho a^3$

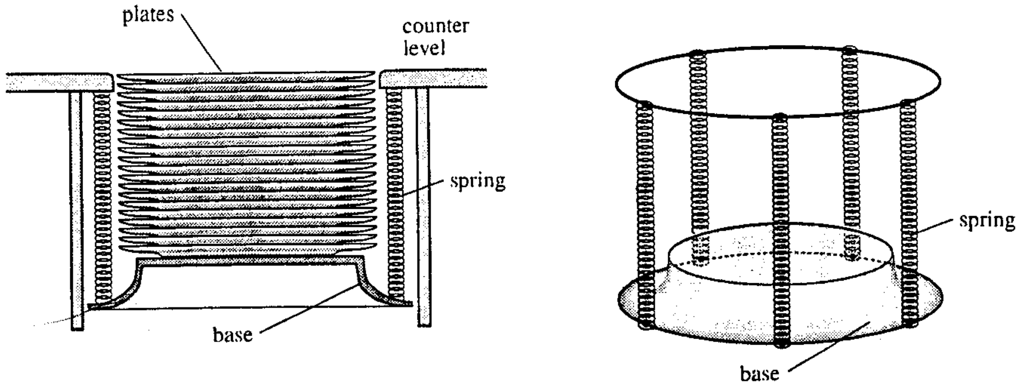
so the MoI of the segment is $\frac{2}{5}Ma^2$ as required.

[Quite a good exercise is to use the segment as a compound pendulum rotating about the same axis. To do this, we need the position of the centre of mass; this is in the plane of symmetry through the straight edge and a distance $\frac{3\pi a \sin\alpha}{16\alpha}$ from it.]

The plate stack

In many self-service cafes, plates are stored in a stack where the top plate is at counter level however many identical plates are in the stack.

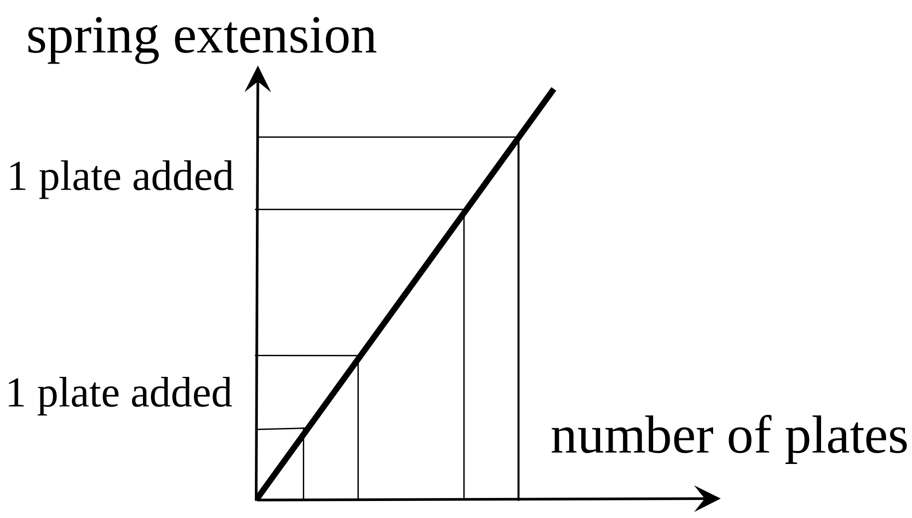
This may be achieved very simply by suspending the plates in a rack supported by suitable springs, as shown in the diagram.



All that is necessary is that the springs have a combined stiffness so that when one plate is added, the springs extend by an amount equal to the increase in height of the stack.

Hooke's Law states that the total extension of a spring or elastic string and the tension in it are directly proportional.

Because this is a linear relationship, addition of a plate to the stack will increase the total tension in springs by the weight of the plate and cause exactly the *same* extension in the springs *however many* plates are already in the stack.



Suppose the combined stiffness of the springs is k N m^{-1}, the weight of each plate is w N and the addition of a plate increases the height of the stack by y m. We have, by HL, that the extension of the springs, e m, caused by adding a plate is given by $w = ke$ so $e = \frac{w}{k}$ and we need to adjust the springs to make e as close as possible to y.

In practice, this seems to be done by changing the number of springs. Of course, one doesn't accept solutions with $e > y$ as this will lead to the top of the pile getting further below the counter as more plates are added to the stack.

Many students find this example counter-intuitive. They 'feel' that just one plate could be above the counter level but that a stack of, say, twenty plates could be below it.