Teaching trigonometry at A level

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Churchill's thoughts

We were arrived in an 'Alice-in-Wonderland' world, at the portals of which stood 'A Quadratic Equation.' This with a strange grimace pointed the way to the Theory of Indices, which again handed on the intruder to the full rigours of the Binomial Theorem. Further dim chambers lighted by sullen, sulphurous fires were reputed to contain a dragon called the 'Differential Calculus.' But this monster was beyond the bounds appointed by the Civil Service Commissioners who regulated this stage of Pilgrim's heavy journey. We turned aside, not indeed to the uplands of the Delectable Mountains, but into a strange corridor of things like anagrams and acrostics called Sines, Cosines and Tangents.

Apparently they were very important, especially when multiplied by each other, or by themselves! They had also this merit—you could learn many of their evolutions off by heart. There was a question in my third and last Examination about these Cosines and Tangents in a highly square-rooted condition which must have been decisive upon the whole of my after life. It was a problem. But luckily I had seen its ugly face only a few days before and recognised it at first sight.

My assumptions

- You are teaching, or are about to teach, Core Maths at AS and/or A2
- You are teaching the MEI A level specification (it doesn't matter if you aren't)
- You have wonderful ideas about how to teach trigonometry!
- You like to do mathematics!

Let's do some maths

- Borchardt and Perrott, "A New Trigonometry for Schools" (1904)
- Examples VI, no.38: prove that

$$\frac{\cot\theta+\operatorname{cosec}\theta}{\operatorname{cosec}\theta-\cot\theta}=\frac{\sin^2\theta}{(1-\cos\theta)^2}$$

Let's do some maths

- Did you enjoy that?
- What was the point of it?
- How would you introduce trigonometry to, say, Year 9?
- Selly Oak Hospital Can Always Handle The Occasional Accident
- Storing Old Hats Till Old Age Causes Amazing Hairiness

Introducing trigonometry

- What I think is the Eastern European approach
- Borchardt and Perrott do this: "practical" applications (i.e. finding sides) come in Chapter 5, while identities appear in Chapter 3
- Churchill obviously learned it this way
- But see Sherlock Holmes!

The Musgrave Ritual

"When my old tutor used to give me an exercise in trigonometry, it always took the shape of measuring heights. When I was a lad I worked out every tree and building in the estate."

Teaching trig at A level

- What do your students think of trig?
- Do they enjoy it?
- What do they get wrong?

Themes in A level trig

- Solving triangles
  - including area of a triangle
- "Definitions" and graphs
  - incl. transformations of graphs
- Solving equations and using identities
  - incl. applications to parametric equations
- Circular measure
- Calculus of trig functions (not considered here)

Themes in A level trig

- Are these the right themes?
- Which are the most important?
- What about modelling periodic phenomena?

Where these topics come

Solving triangles

- GCSE (refs to OCR J567)
  - Right-angle trig (HSG4)
  - Pythagoras and trig in 3d contexts (HGG2)
  - Area formula; sine and cosine rule in 2d & 3d (HGG3)
- Additional Maths (OCR 6993)
  - Right-angle trig
  - Sine/cosine rules
  - Apply trig to triangles with any angles/to 2d & 3d problems

Where these topics come

Solving triangles

- AS (MEI Core 2)
  - Right-angle trig (C2t1)
  - Area formula (C2t8)
  - Sine & cosine rules (C2t9)

Where these topics come

"Definitions" and graphs

- GCSE
  - Draw, sketch and recognise…the trig functions y = sin x and y = cos x for any angle (HGA5)
  - ...and use to solve equations (HGA5)
  - (transformations of graphs (HGA6))
- Additional Maths
  - Be able to use the definitions of sin x, cos x and tan x for any angle

Where these topics come

"Definitions" and graphs

- AS (Core 2)
  - Use definitions of sin x and cos x for any angle (C2t2)
  - Know graphs, symmetries and periodicity (C2t3)
  - Know exact values for special angles (C2t4)
  - Stretches (C2C2 – translations are in C1)
- A2 (Core 3)
  - Effect of combined transformations (C3f2, f4)
  - Understand arcsin, arccos and arctan, graphs and domains (C3f7)

Where these topics come

"Definitions" and graphs

- A2 (Core 4)
  - Definitions of sec, cosec, cot (C4t1)
  - Graphs of sec, cosec, cot (C4t2)

Where these topics come

Solving equations and using identities

- GCSE
  - Graphs...and use to solve equations (HGA5)
- Additional Maths
  - Solve simple trig equations in given intervals
  - Know and use tan x = sin x/cos x and Pythagorean identity
- AS (Core 2)
  - Identities as in Add Maths (C2t5, t6)
  - Solve simple trig equations in given intervals (C2t7)

Where these topics come

Solving equations and using identities

- A2 (Core 4)
  - Identities involving sec, cot, cosec (C4t3)
  - Compound angle formulae (C4t4)
  - Double angle formulae (C4t5)
  - Equations involving these identities (C4t6)
  - The form r sin(θ ± α) and applications (C4t7)
  - Applications to parametric equations (C4g2)

Where these topics come

Circular measure

- AS (Core 2)
  - Definition of a radian: radian/degree conversion (C2t10)
  - Arc length, area of sector (C2t11)

Where these topics come

- Are these topics in the right places? Your thoughts?
- MEI (and Michael Gove?) are revising the specification…there's a move to bring sec, cosec and cot into AS

AS Level

Solving triangles

- Nothing really new at AS after Higher Level GCSE
- I set the sine rule, cosine rule and area of triangle as self-study
- In MEI, often a context-based question in Section B, e.g. S11 Q13 (next slide: Q also covers radians, arcs and sectors)
- Candidates usually do these questions very well, although they can be caught out by rounding and don't like working in radians

C2 S11 Q13

Fig. 13.1 shows a greenhouse which is built against a wall.

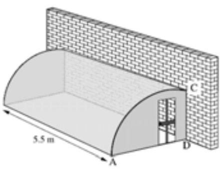

Fig. 13.1

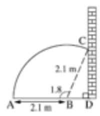

Fig. 13.2

The greenhouse is a prism of length 5.5 m. The curve AC is an arc of a circle with centre B and radius 2.1 m, as shown in Fig. 13.2. The sector angle ABC is 1.8 radians and ABD is a straight line. The curved surface of the greenhouse is covered in polythene.

(i) Find the length of the arc AC and hence find the area of polythene required for the curved surface of the greenhouse. **[4]**

(ii) Calculate the length BD. **[3]**

(iii) Calculate the volume of the greenhouse. **[5]**

C2 S11 Q13

13 (i)	Those who worked in radians often scored full marks, although some candidates just found the area of the sector as if on autopilot and ignored the length of the greenhouse. A good number of candidates preferred to work in degrees, and a proportion of these inevitably made errors in the conversion process and lost marks. A few candidates converted θ to degrees and then multiplied it by 2.1.
13 (ii)	This should have been a straightforward exercise in finding a length in a right angled triangle, but many went astray because they were unable to find angle *CBD* correctly. Others chose a convoluted method such as the Sine Rule or finding CD first and using Pythagoras, and then made algebraic or arithmetic slips. It was the exception, rather than the norm, to see the expected approach of $2.1\times\cos(\pi - 1.8)$ leading to the correct answer. Some candidates lost the accuracy mark because their calculator was in the wrong mode.
13 (iii)	Most candidates were able to find the area of the sector correctly, but too many candidates made finding the area of the triangle far more complicated than necessary. The usual approach was to find *CD* and then use ½base × height, and marks were often lost due to mistakes in applying the Pythagoras formula. As with part (ii), the expected approach ($\frac{1}{2} \times 2.1 \times BD \times \sin(\pi - 1.8)$) was the exception rather than the norm. Most realised the need to sum the two areas and then multiply by 5.5.

AS Level

Definitions and graphs

- How would you **define** sin *x*, cos *x*, tan *x*?
- Autograph Extras has a nice demonstration
- The graphs are well known, but candidates have found describing transformations difficult: they struggle to use precise language
- They have also struggled with the idea of an **exact** value
- Modern calculators can give exact forms: is this good?

C2 S11 Q10

The nth term, t_n, of a sequence is given by

$$t_n = \sin(\theta + 180n)^\circ.$$

Express t_1 and t_2 in terms of $\sin\theta^\circ$. **[2]**

Very few candidates understood this question, and convincing fully correct answers were seldom seen. The most common approach was $\sin(\theta + 180) = \sin\theta + \sin180$ etc., which did not score.

C2 S09 Q1

Use an isosceles right-angled triangle to show that $\cos 45° = \frac{1}{\sqrt{2}}$. [2]

There were some good clear answers to this question, but many candidates failed to score both marks. Elementary mistakes with the right angled triangle, such as $1^2 + 1^2 = 2^2$, were common. For the second mark, $\cos\theta = \frac{\text{opp}}{\text{hyp}}$ was seen quite often.

AS Level

Equations and identities

- Focus in MEI C2 is on solving equations
- And it's not done very well!
- Candidates don't know how to find "the other" value, e.g. cos x = -0.3 gives x = 107.5 and then they give 287.5
- Candidates miss roots by dividing through by a trig ratio
- Candidates don't like working in radians
- How do we help improve their work in this area?

C2 S10 Q8

8 Showing your method clearly, solve the equation $4\sin^2\theta = 3 + \cos^2\theta$, for values of θ between 0° and 360°. [5]

Most candidates realised that $1 - \sin^2\theta$ or $1 - \cos^2\theta$ needed to be substituted, many had problems rearranging the equation in a suitable form. Many who obtained, say, $5\cos^2\theta = 1$ failed to take the square root and found $\cos^{-1}0.2$ – a few of these then went on to square root the angle. Approximately one third of candidates did take the square root correctly at this stage; of these approximately one quarter dealt with the negative root. Some candidates failed to score at all, either because they substituted $\cos^2\theta - 1$ or because they divided through by $\cos\theta$ and fudged the algebra.

C2 S09 Q7

7 Show that the equation $4\cos^2\theta = 4 - \sin\theta$ may be written in the form

$$4\sin^2\theta - \sin\theta = 0.$$

Hence solve the equation $4\cos^2\theta = 4 - \sin\theta$ for $0° \leqslant \theta \leqslant 180°$. [5]

Most candidates knew that the substitution $\cos^2\theta = 1 - \sin^2\theta$ was expected, and were able to show at least one correct step in obtaining the required result. A few incurred a penalty by failing to make clear what they were doing. Some weak candidates simply manipulated the original expression, "went round the houses" and ended up back at the start point. Most candidates went on to obtain 14.47° and 165.53°, but omitted either 0° or 180° - or in some instances both.

C2 W09 Q4

4 Solve the equation $\sin 2x = -0.5$ for $0° < x < 180°$. [3]

4) This question was not done well. An initial step for most was "sin$x$ = - 0.25", which resulted in no marks. Often the better candidates failed to obtain a complete solution. 15° was often presented as part of the final answer.

Moral: trig functions are non-linear

- Get them to explore sin 2x and sin(x + y) etc. and disprove a few things?

AS Level

<u>Circular measure</u>

- How do you motivate the use of radians?
- $\sin x \approx x$ for "small" x in radians
- Some convert everything to degrees, then back again
- MEI C2: questions in Section A are often straightforward and done very well
- Contextualised and problem-solving style questions are less well done

C2 W10 Q4

4 A sector of a circle has area $8.45\,\text{cm}^2$ and sector angle 0.4 radians. Calculate the radius of the sector. **[3]**

This question was done very well indeed. Only a few candidates used $r\theta$ or $r\theta^2$; fewer still failed to manipulate the correct formula to obtain the correct answer. Even the majority of those who converted to degrees earned full marks by using their calculators effectively. A very small minority thought the angle was 0.4π radians.

C2 W11 Q9

Charles has a slice of cake; its cross-section is a sector of a circle, as shown in Fig. 9. The radius is r cm and the sector angle is $\frac{\pi}{6}$ radians.

He wants to give half of the slice to Jan. He makes a cut across the sector as shown.

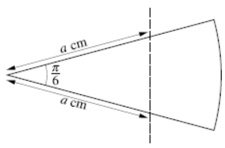

Fig. 9

Show that when they each have half the slice, $a = r\sqrt{\frac{\pi}{6}}$. **[4]**

C2 W11 Q9

Many candidates earned a mark by stating correctly the area of the sector. A few were able to also give the correct formula for the area of the triangle (but a surprisingly large number could not), but only the best were able to deal with $\sin(\pi/6)$ and equate this with half the area of the sector. There were many fruitless attempts to "fudge" the given answer, often based on using the length a as the area.

A2 Level

<u>Definitions and graphs in Core 3</u>

- Combined transformations (beyond GCSE?): opportunities for modelling, e.g. length of daylight
- Language and ideas of functions: inverses and their names
- $\arcsin x$ is sometimes thought to be $1/\sin x$
- Domain and range "baffle" candidates
- Sometimes comes into proof questions

C3 W10 Q7

7 Given that $\arcsin x = \arccos y$, prove that $x^2 + y^2 = 1$. [Hint: let $\arcsin x = \theta$.] **[3]**

This proof question was also generally not well done, despite the hint. Getting from $\theta = \arccos x$ to $x = \cos\theta$ appeared to be considerably harder than might be expected. Some candidates correctly deduced that $x = \cos\theta$ and $y = \sin\theta$ but failed to convince by appearing to argue that $x^2 + y^2 = 1 \Rightarrow \cos^2\theta + \sin^2\theta = 1$. Indeed, many candidates seem to fail to understand the concept of direction of argument in proof. Common errors seen were $\arcsin x = 1/\sin x$ and $\cos x = \sin y$.

C3 S10 Q6

6 The function $f(x)$ is defined by

$$f(x) = 1 + 2\sin 3x, \quad -\frac{\pi}{6} \leqslant x \leqslant \frac{\pi}{6}.$$

You are given that this function has an inverse, $f^{-1}(x)$.

Find $f^{-1}(x)$ and its domain. **[6]**

Finding the inverse function was well done, with the commonest error mishandling the step from $\sin 3y = (x - 1)/2$ to get $y = \arcsin (x - 1)/6$. However, domains and ranges often seem to baffle candidates, and the domain given here was often incorrect.

A2 Level

Equations and identities in Core 4

- Lots of ideas: some remembering needed!
- Reciprocal trig functions: what is a "secant"? Solve cot x = 0…
- Identities: derive from Pythagoras/find a diagram
- Solving equations e.g. cot $2x$ = 3: how many ways can we find?
- Equations cause same problems as in C2, e.g. omission of roots and rounding errors
- $R \sin(\theta \pm \alpha)$ method well known: how do you teach it? Applications to e.g. max/min can cause problems

A2 Level

Equations and identities in Core 4

- Compound angle formulae: Eric the piece of cardboard
- Derive double angle formulae from these (or learn? cos 2θ often "remembered" incorrectly)
- Identities sometimes needed to change parametric equations to Cartesian (capital C?)
- Sometimes trig is the basis for a big contextual question in Section B: these are usually done well
- Weak algebraic skills cause problems

C4 S11 Q5

5 Solve the equation $\operatorname{cosec}^2 \theta = 1 + 2\cot\theta$, for $-180° \leqslant \theta \leqslant 180°$. **[6]**

This was a harder trigonometry question than on recent papers and few candidates scored the full six marks.

When using the first method, many failed to use the correct trigonometrical identity. Then, a common error was to 'lose' one of the factors eg $\cot^2\theta - 2\cot\theta = 0$, $\cot^2\theta = 2\cot\theta$, $\cot\theta = 2$. Others obtained $\cot\theta = 0$ but did not find the solutions 90° and -90° (considering the graph of $\cot\theta$ or using $\cot\theta = \cos\theta/\sin\theta$ may have helped), often 180°, 0°, and -180° being given.

Those that chose to express their equation in terms of $\sin\theta$ and $\cos\theta$ from the outset usually obtained $\sin^2\theta + 2\sin\theta\cos\theta - 1 = 0$. Much then depended upon them using Pythagoras to obtain $2\sin\theta\cos\theta - \cos^2\theta = 0$ and factorising. Once again, $2\sin\theta = \cos\theta$ was often seen and the $\cos\theta = 0$ was forgotten.

Candidates would be advised to factorise instead of cancelling.

C4 S11 Q3

3 Express $2\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants to be determined, and $0 < \alpha < \frac{1}{2}\pi$.

Hence write down the greatest and least possible values of $1 + 2\sin\theta - 3\cos\theta$. **[6]**

The first four marks were usually obtained. Candidates seemed to be well prepared for the 'R-method' and the number achieving full marks has improved. The most common error was the use of degrees instead of radians.

The final part, when finding the greatest and least possible values, caused some confusion. Whilst there were many completely correct solutions some omitted the 1 and just gave $\pm\sqrt{13}$ and many others tried to find angles from say solving $\sqrt{13}\sin(\theta - 0.983) = -1$ for θ.

C4 S10 Q3

3 The parametric equations of a curve are

$$x = \cos 2\theta, \quad y = \sin\theta\cos\theta \quad \text{for } 0 \leqslant \theta < \pi.$$

Show that the cartesian equation of the curve is $x^2 + 4y^2 = 1$.

Sketch the curve. **[5]**

This was the least successfully answered question. A wide variety of methods were seen. Some candidates used $\sin 2\theta = 2\sin\theta\cos\theta$ and substituted $2y = \sin 2\theta$ in $\sin^2 2\theta + \cos^2 2\theta = 1$. More commonly, others chose longer methods involving changing $\cos 2\theta$ to $\cos^2\theta - \sin^2\theta$, squaring, and adding to $4\sin^2\theta\cos^2\theta$. Often this work was inaccurate. $(\cos^2\theta - \sin^2\theta)^2 = \cos^4\theta + \sin^4\theta$ and similar errors when squaring were seen. For those with this approach, many failed to proceed further instead of using $\sin^2\theta + \cos^2\theta = 1$ and substituting. Many attempts involved many false starts at trigonometrical equations without making real progress.

Relatively few candidates drew an ellipse. This part was frequently omitted. Some drew a circle (which was allowed if the axes had different scales) or part of an ellipse and there were a wide variety of other incorrect graphs.

C4 W11 Q8

Fig. 8 shows a searchlight, mounted at a point A, 5 metres above level ground. Its beam is in the shape of a cone with axis AC, where C is on the ground. AC is angled at α to the vertical. The beam produces an oval-shaped area of light on the ground, of length DE. The width of the oval at C is GF. Angles DAC, EAC, FAC and GAC are all β.

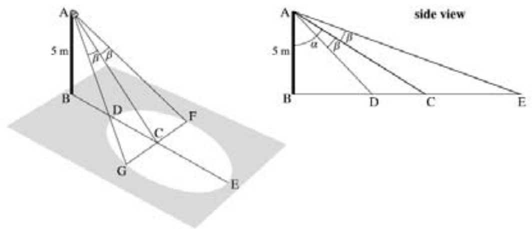

Fig. 8

In the following, all lengths are in metres.

(i) Find AC in terms of α, and hence show that $GF = 10\sec\alpha\tan\beta$. **[3]**

C4 W11 Q8

(ii) Show that $CE = 5(\tan(\alpha+\beta) - \tan\alpha)$.

Hence show that $CE = \dfrac{5\tan\beta\sec^2\alpha}{1-\tan\alpha\tan\beta}$. **[5]**

Similarly, it can be shown that $CD = \dfrac{5\tan\beta\sec^2\alpha}{1+\tan\alpha\tan\beta}$. [You are **not** required to derive this result.]

You are now given that $\alpha = 45°$ and that $\tan\beta = t$.

(iii) Find CE and CD in terms of t. Hence show that $DE = \dfrac{20t}{1-t^2}$. **[5]**

(iv) Show that $GF = 10\sqrt{2}t$. **[2]**

For a certain value of β, DE = 2GF.

(v) Show that $t^2 = 1 - \dfrac{1}{\sqrt{2}}$.

Hence find this value of β. **[3]**

Further topics

Should these be in A level?

- Small angles
- Factor formulae
- Power series expansions
  - How do calculators find trig ratios?
- Integrals involving trig substitutions/inverse trig functions
- The $t = \tan x/2$ substitution

Further Maths

Further Pure 1

- The argument of a complex number
- Rotation matrices

Further Pure 2

- Maclaurin series
- Polar co-ordinates
- De Moivre's Theorem
- Sin/cos/tan of multiple angles

Further Maths

Further Pure 2

- $C + jS$ method of summing (often trigonometric) series
- Calculus of inverse trigonometric functions

Thank you for participating

"Our maths brain lies often awake in his bed,
Doing logs to ten places and trig in his head"
(Alan Turing's verse in a House song)

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