MEI

Innovators in Mathematics Education

Mathematics in Education and Industry

50 years at the forefront of Mathematics Education

Ideas for a Further Mathematics classroom

WHY BOTHER WITH VECTOR EQUATIONS

Starter…

Particle A is projected from the position $(3,-1,2)$ with velocity $(-1,6,2)$.

At the same instant, particle B is projected from the position $(5,5,4)$ with velocity $(-2,3,1)$.

Do the particles collide?

Vectors make things easier…

$$\mathbf{r}_A = \mathbf{s}_A + \mathbf{u}_A t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{r}_B = \mathbf{s}_B + \mathbf{u}_B t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s}_A + \mathbf{u}_A t + \frac{1}{2}\mathbf{a}t^2 = \mathbf{s}_B + \mathbf{u}_B t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s}_A + \mathbf{u}_A t = \mathbf{s}_B + \mathbf{u}_B t$$

$$\mathbf{s}_A - \mathbf{s}_B = \mathbf{u}_B t - \mathbf{u}_A t$$

$$\mathbf{s}_A - \mathbf{s}_B = -t(\mathbf{u}_A - \mathbf{u}_B)$$

Particle A is projected from the position $(3, -1, 2)$ with velocity $(-1, 6, 2)$.

At the same instant, particle B is projected from the position $(5, 5, 4)$ with velocity $(-2, 3, 1)$.

Do the particles collide?

$$\boldsymbol{s}_A - \boldsymbol{s}_B = -2\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$$

$$\mathbf{u}_A - \mathbf{u}_B = 1\mathbf{i} + 3\mathbf{j} + 1\mathbf{k}$$

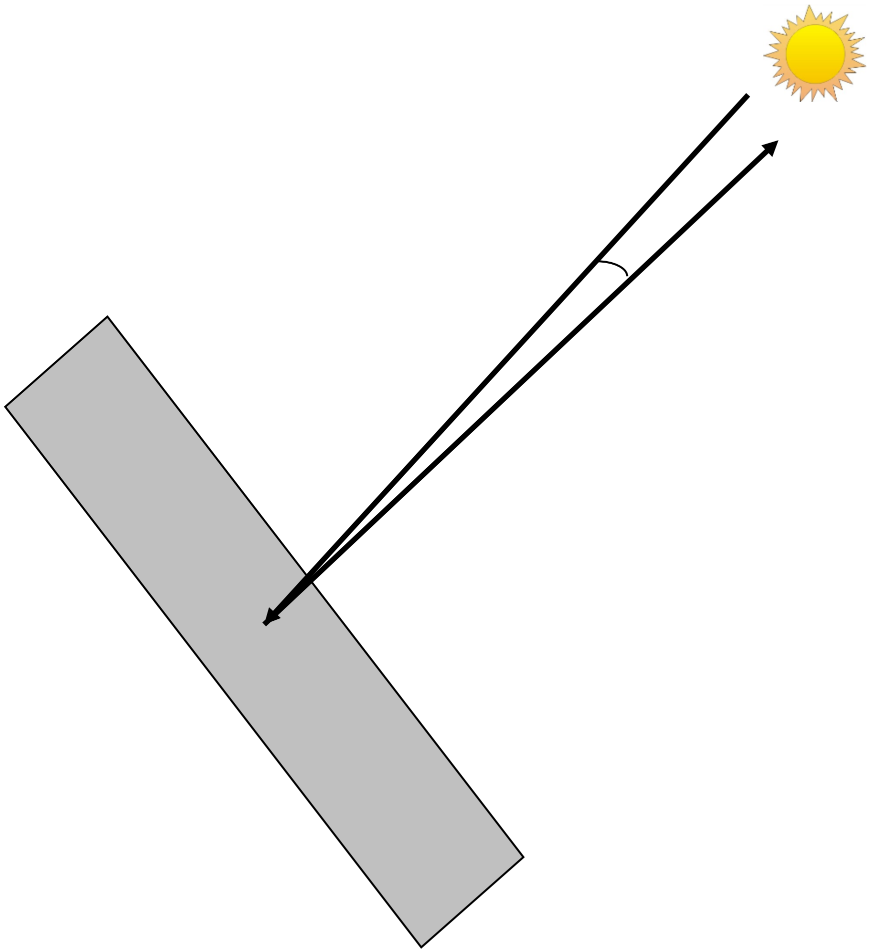
The vector equation of a plane

$$\mathbf{r}.\mathbf{n} = d$$

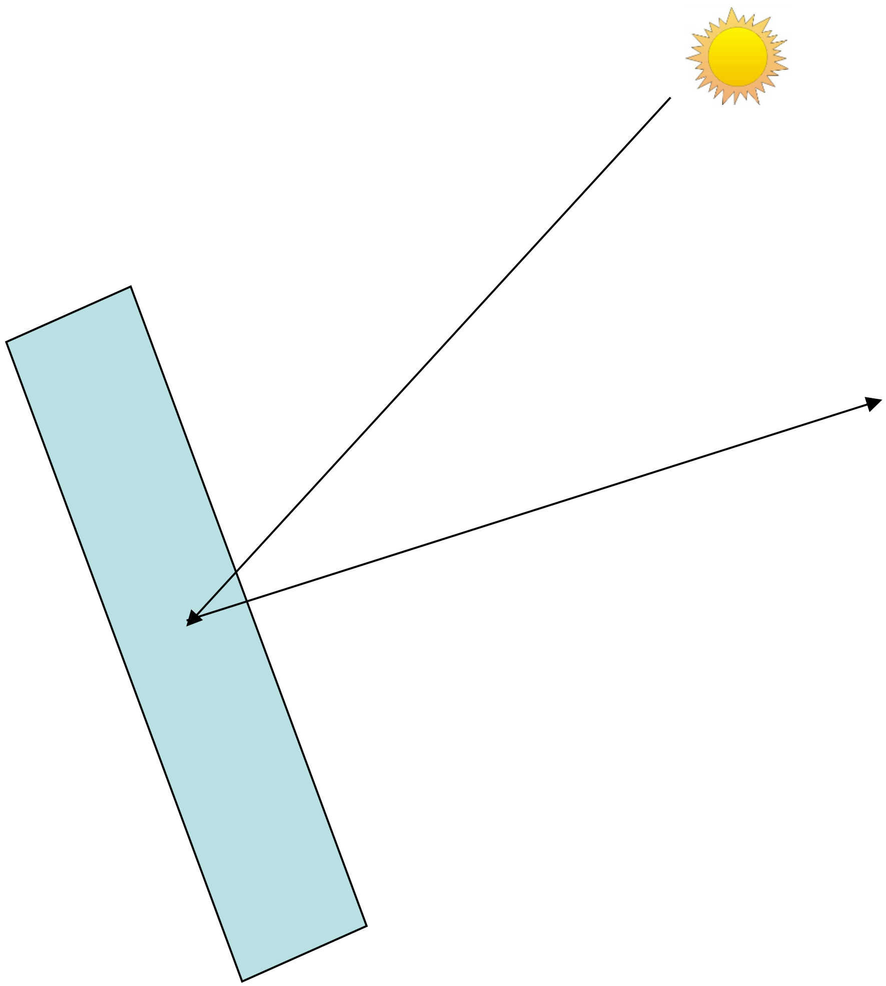
Find the angle between the plane $\mathbf{r}.\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = 2$

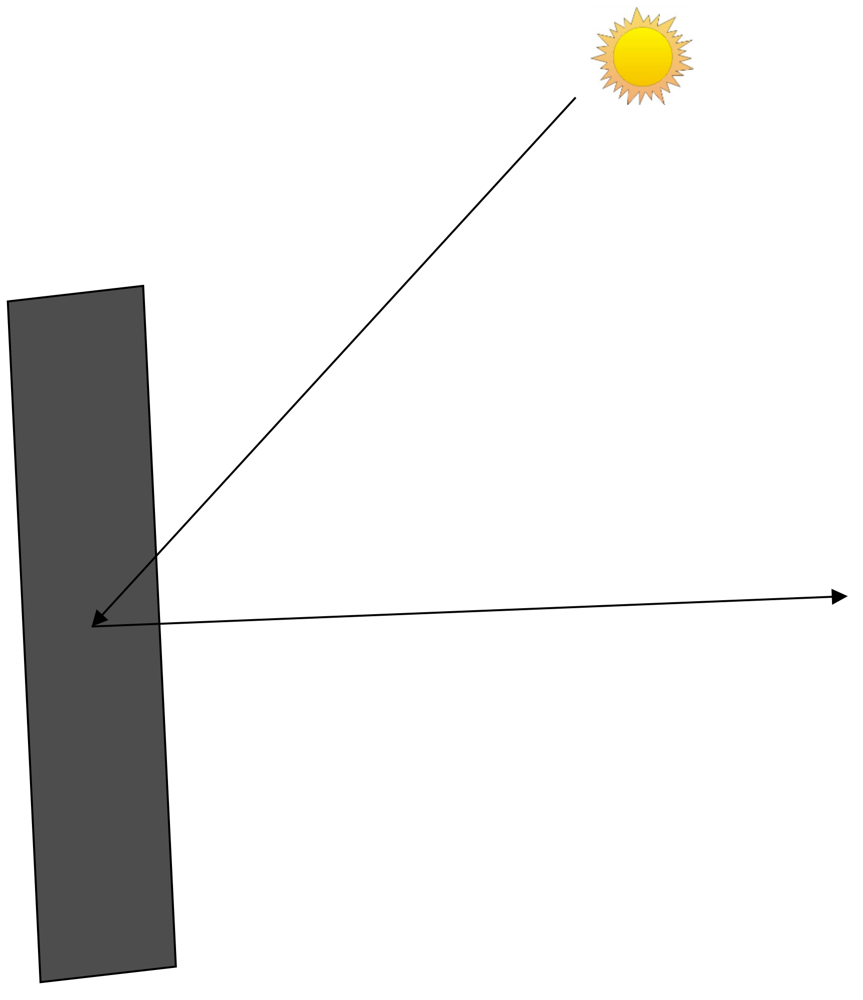
and the line $\mathbf{r} = \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

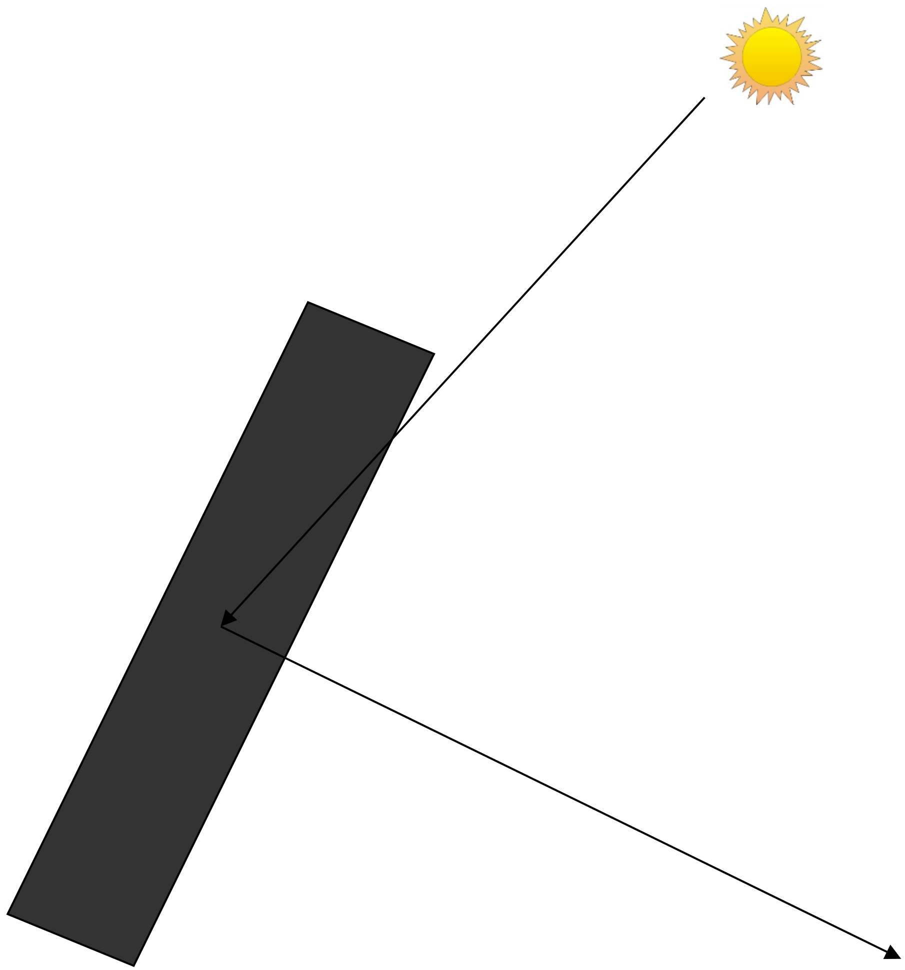
Dynamic lighting

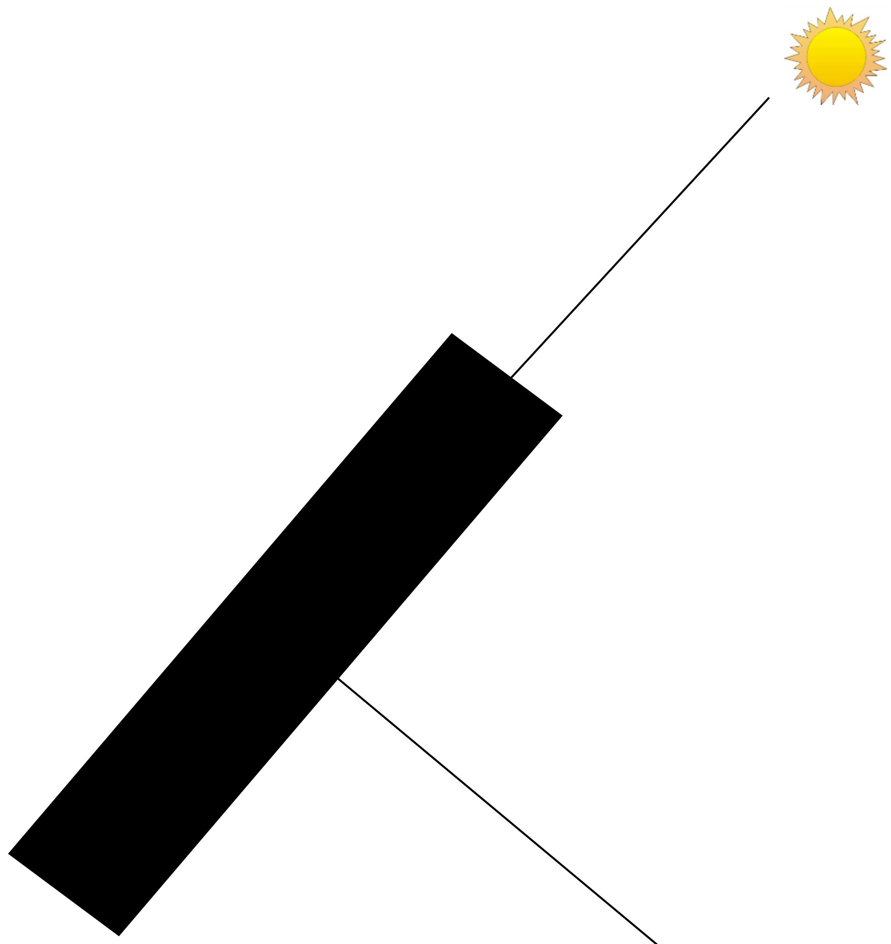


Angle between normal vector and light source determines how light the surface should appear









The dot product and computer games..

The dot product is the Swiss army knife of computer games

It has hundreds of uses

Sometimes new ones are found accidentally

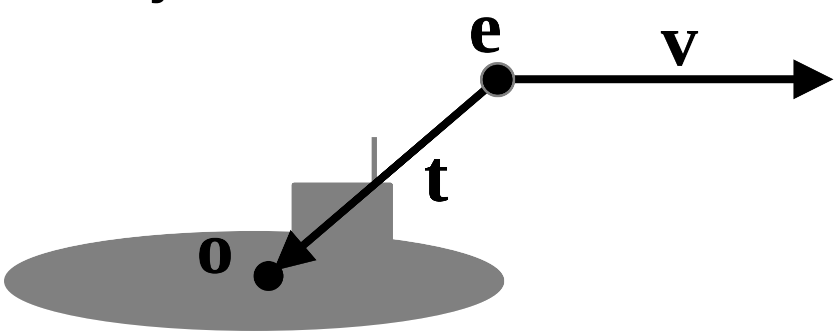
Angles: View Testing

Simple view culling

View vector **v** and vector **t** to object in scene

$$\mathbf{t} = \mathbf{o} - \mathbf{e}$$

If $\mathbf{v} \bullet \mathbf{t} < 0$, object behind us, cull



Angles: Collision Response

Have normal $\mathbf{n}$ (from object A to object B), relative velocity $\mathbf{v}_A - \mathbf{v}_B$

Three cases of contact:

- Separating

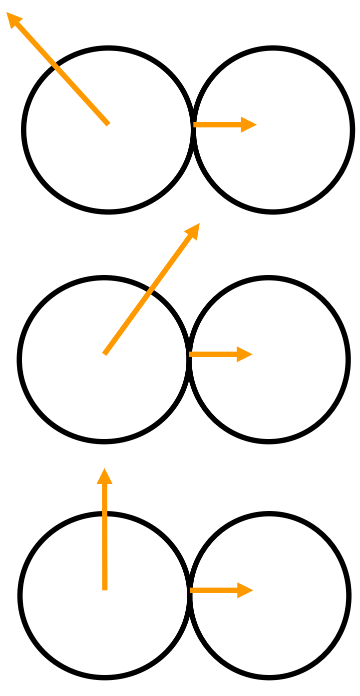
$$(\mathbf{v}_A - \mathbf{v}_B) \bullet \mathbf{n} < 0$$

- Colliding

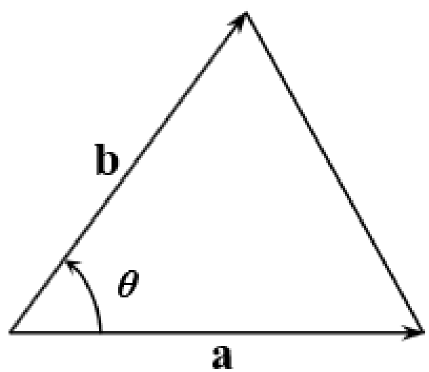
$$(\mathbf{v}_A - \mathbf{v}_B) \bullet \mathbf{n} > 0$$

- Resting

$$(\mathbf{v}_A - \mathbf{v}_B) \bullet \mathbf{n} = 0$$

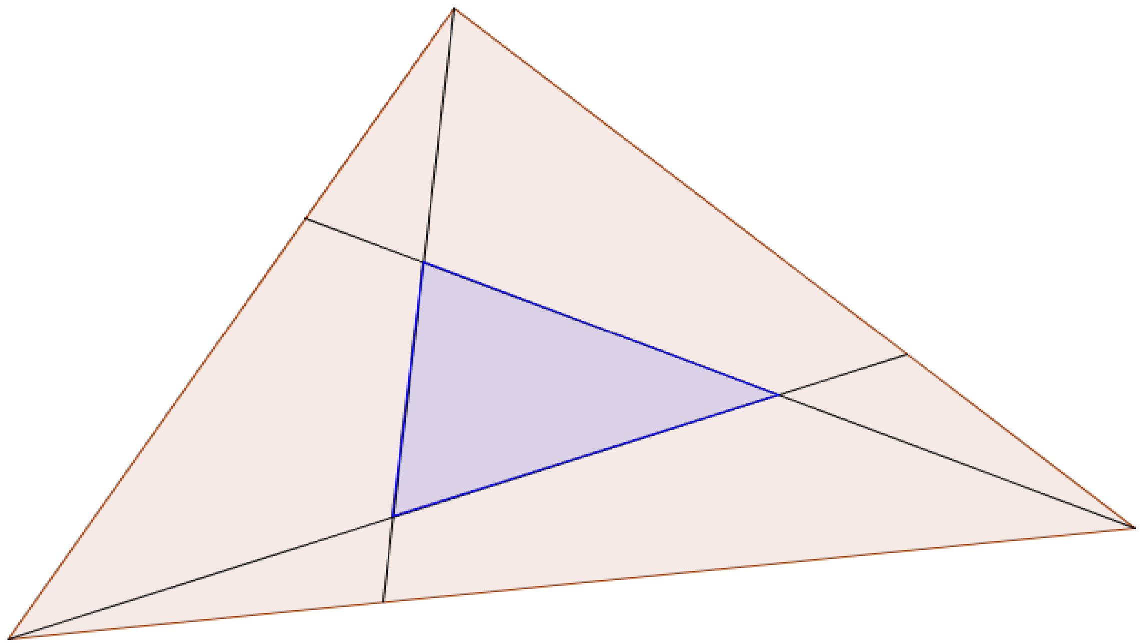


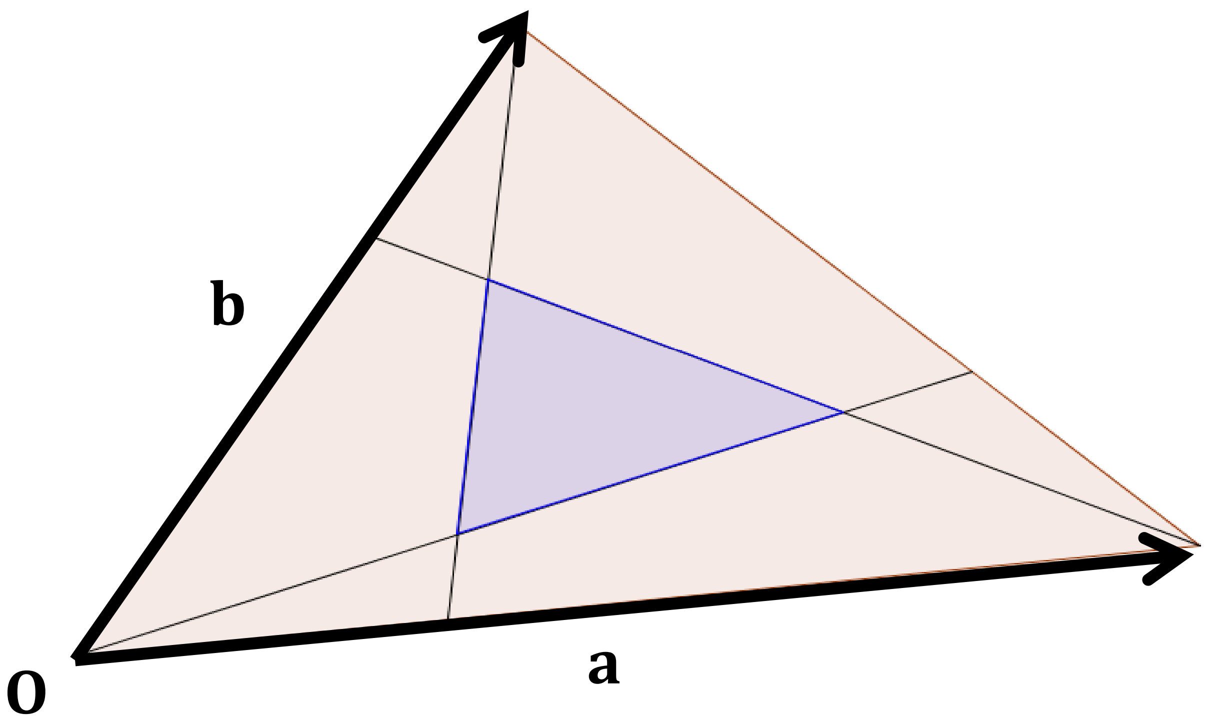
Vector product: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}}$

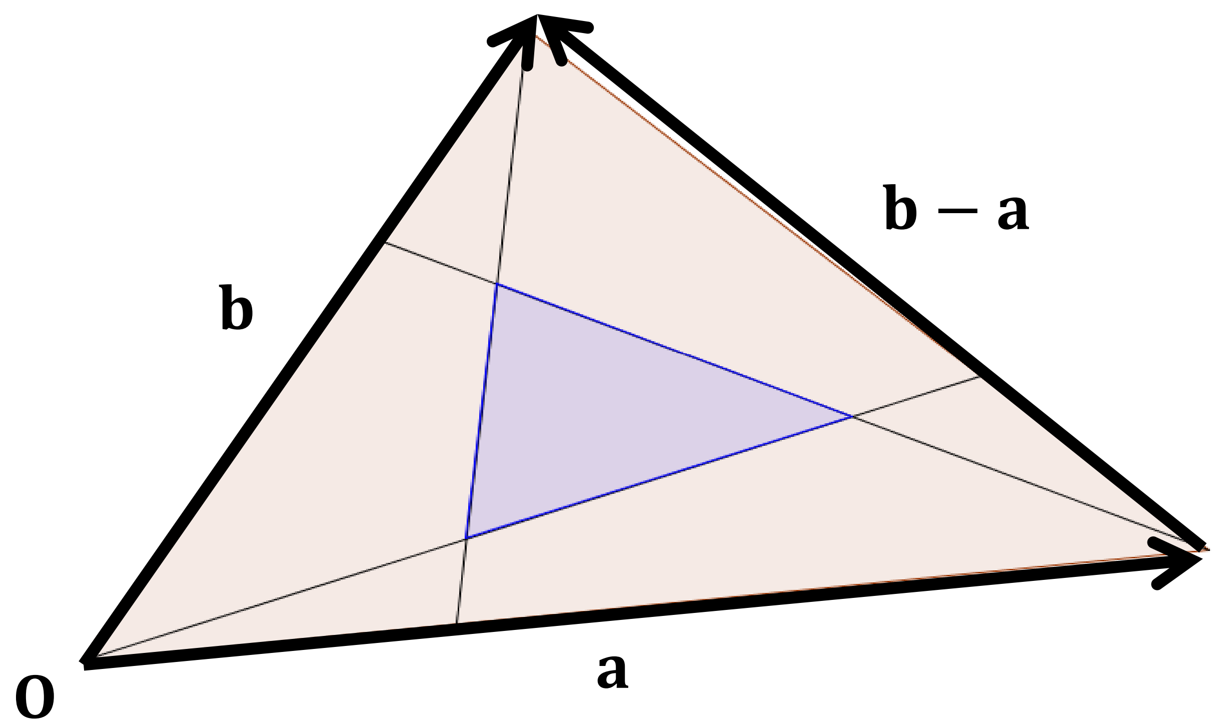


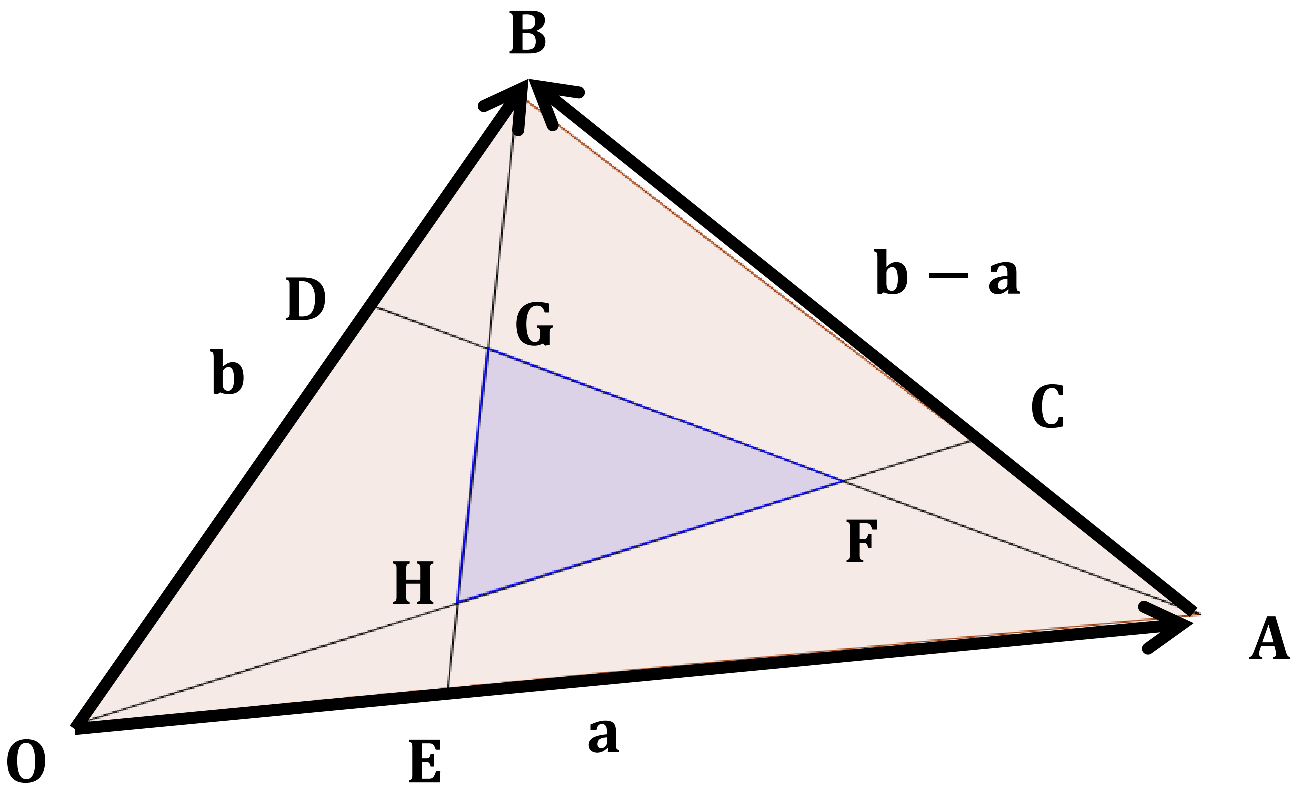
Area $= \frac{1}{2} ab \sin\theta = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

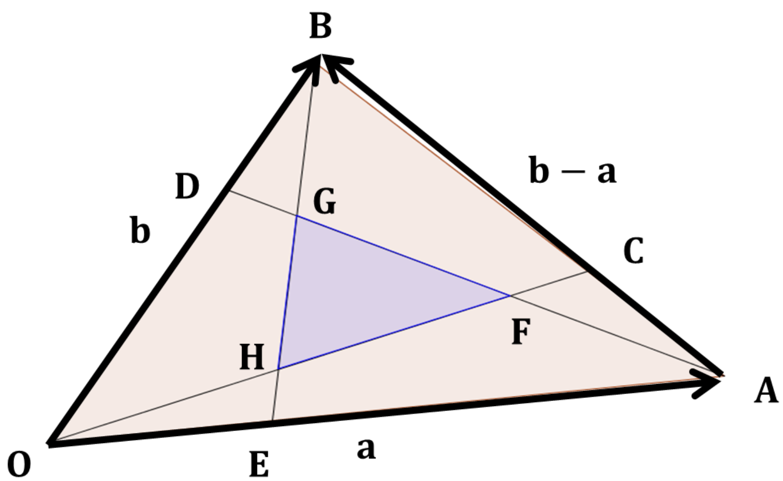
The area of a triangle is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$




b
a
O


b − a
b
a
O


B
b − a
D
G
b
C
F
H
A
a
O
E

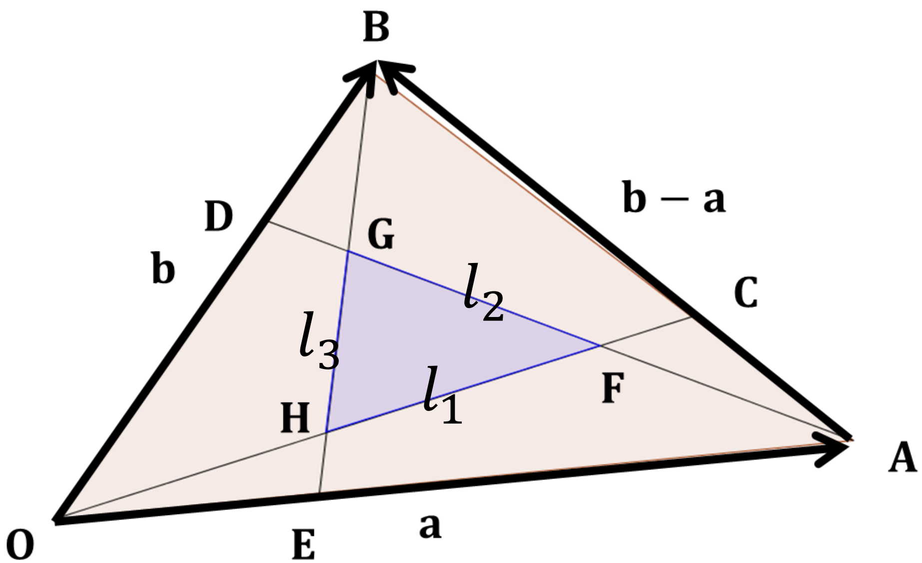


Area of OAB$= \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

$$\overrightarrow{OC} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$$

$$\overrightarrow{AD} = \mathbf{b} - \mathbf{a} - \frac{1}{3}\mathbf{b} = -\mathbf{a} + \frac{2}{3}\mathbf{b}$$

$$\overrightarrow{BE} = -\mathbf{b} + \frac{1}{3}\mathbf{a} = \frac{1}{3}\mathbf{a} - \mathbf{b}$$



$$\overrightarrow{OC} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$$

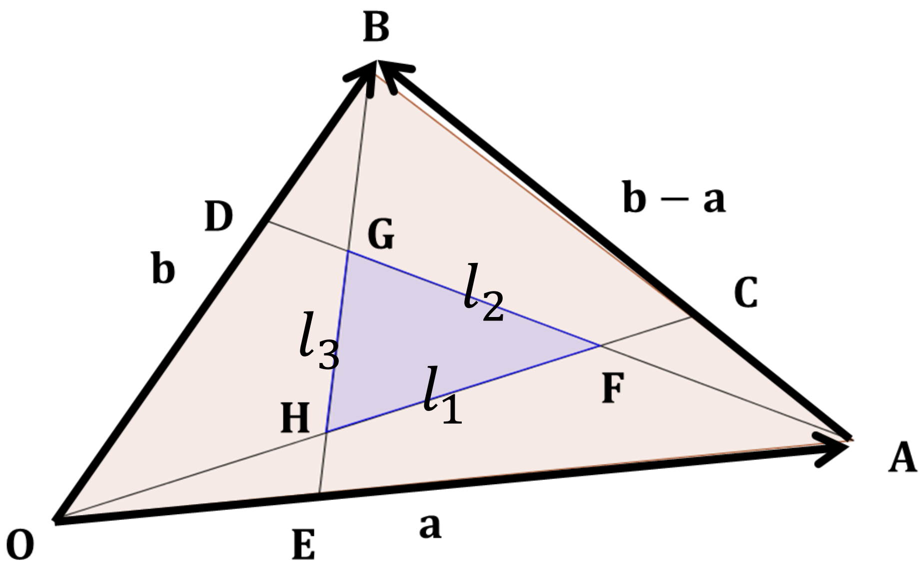
$$\overrightarrow{AD} = -\mathbf{a} + \frac{2}{3}\mathbf{b}$$

$$\overrightarrow{BE} = \frac{1}{3}\mathbf{a} - \mathbf{b}$$

$$l_1: \mathbf{r} = \lambda\left(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)$$

$$l_2: \mathbf{r} = \mathbf{a} + \mu\left(-\mathbf{a} + \frac{2}{3}\mathbf{b}\right)$$

$$l_3: \mathbf{r} = \mathbf{b} + \nu\left(\frac{1}{3}\mathbf{a} - \mathbf{b}\right)$$



$$l_1: \mathbf{r} = \lambda\left(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)$$

$$l_2: \mathbf{r} = \mathbf{a} + \mu\left(-\mathbf{a} + \frac{2}{3}\mathbf{b}\right)$$

$$l_3: \mathbf{r} = \mathbf{b} + \nu\left(\frac{1}{3}\mathbf{a} - \mathbf{b}\right)$$

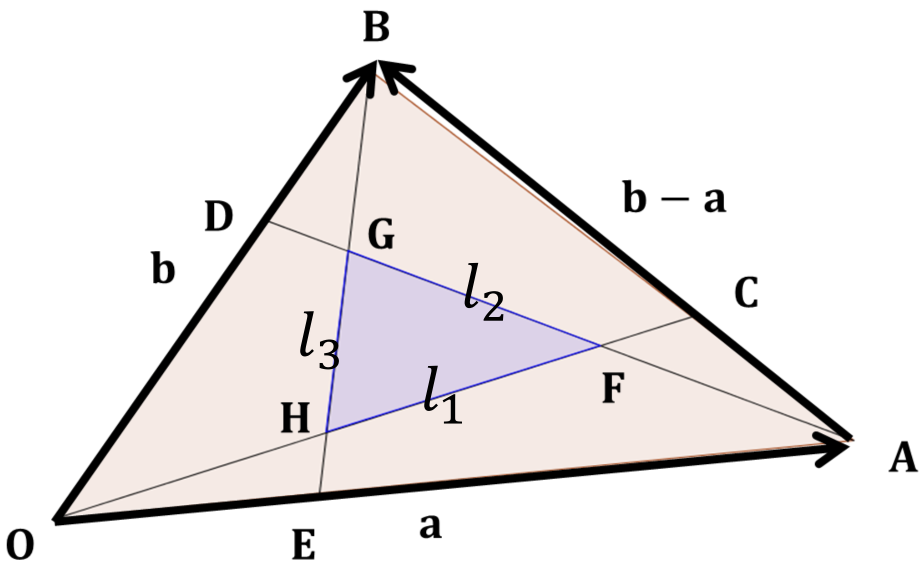
Intersections

F: l_1 and l_2

$$\lambda\left(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right) = \mathbf{a} + \mu\left(-\mathbf{a} + \frac{2}{3}\mathbf{b}\right)$$

$$\frac{2}{3}\lambda = 1 - \mu \qquad \frac{1}{3}\lambda = \frac{2}{3}\mu$$

$$\mu = \frac{3}{7} \qquad \lambda = \frac{6}{7}$$



$$l_1 : \mathbf{r} = \lambda\left(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)$$

$$l_2 : \mathbf{r} = \mathbf{a} + \mu\left(-\mathbf{a} + \frac{2}{3}\mathbf{b}\right)$$

$$l_3 : \mathbf{r} = \mathbf{b} + \nu\left(\frac{1}{3}\mathbf{a} - \mathbf{b}\right)$$

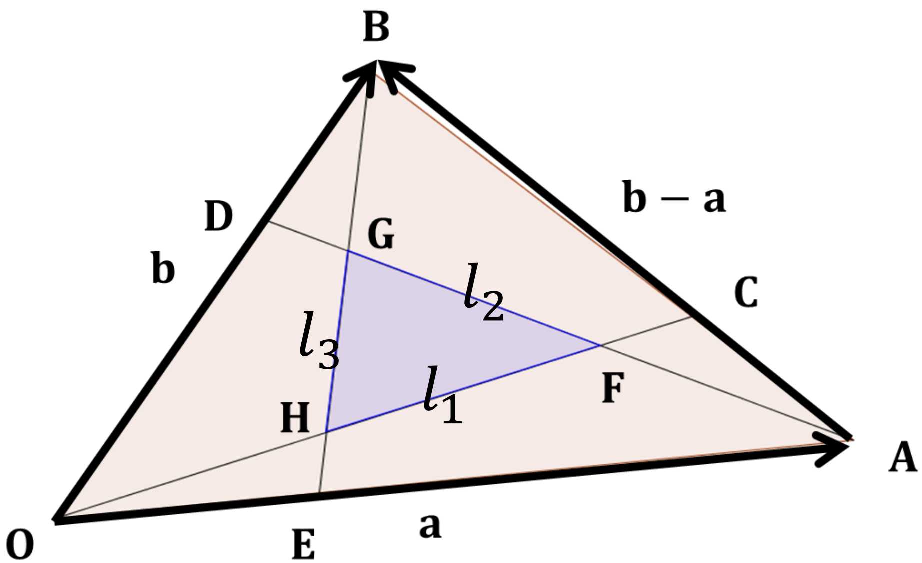
Intersections

G: l_2 and l_3

$$\mathbf{a} + \mu\left(-\mathbf{a} + \frac{2}{3}\mathbf{b}\right) = \mathbf{b} + \nu\left(\frac{1}{3}\mathbf{a} - \mathbf{b}\right)$$

$$1 - \mu = \frac{1}{3}\nu \qquad \frac{2}{3}\mu = 1 - \nu$$

$$\mu = \frac{6}{7} \qquad \nu = \frac{3}{7}$$



$$l_1 : \mathbf{r} = \lambda\left(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)$$

$$l_2 : \mathbf{r} = \mathbf{a} + \mu\left(-\mathbf{a} + \frac{2}{3}\mathbf{b}\right)$$

$$l_3 : \mathbf{r} = \mathbf{b} + \nu\left(\frac{1}{3}\mathbf{a} - \mathbf{b}\right)$$

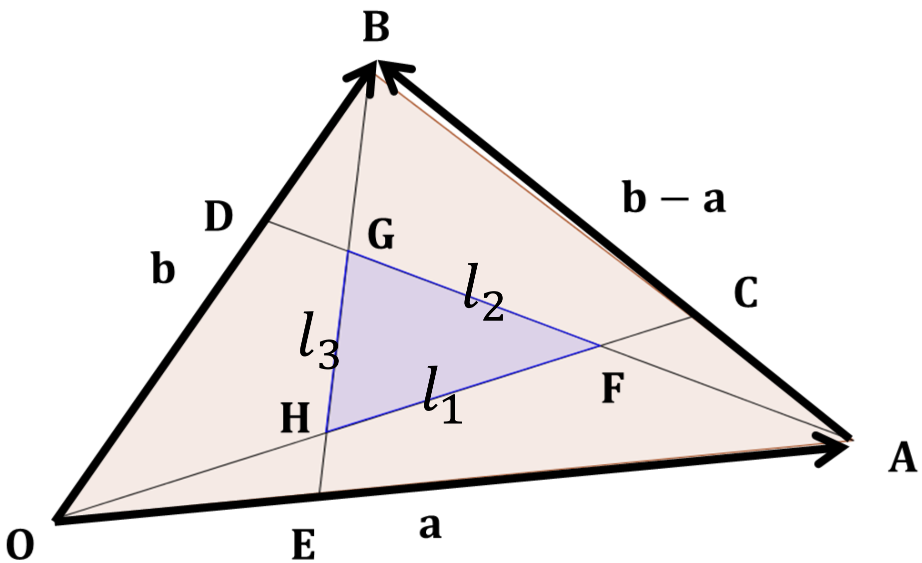
Intersections

H: l_1 and l_3

$$\lambda\left(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right) = \mathbf{b} + \nu\left(\frac{1}{3}\mathbf{a} - \mathbf{b}\right)$$

$$\frac{2}{3}\lambda = \frac{1}{3}\nu \qquad \frac{1}{3}\lambda = 1 - \nu$$

$$\lambda = \frac{3}{7} \qquad \nu = \frac{6}{7}$$



$$\overrightarrow{OF} = \frac{4}{7}\mathbf{a} + \frac{2}{7}\mathbf{b}$$

$$\overrightarrow{OG} = \frac{1}{7}\mathbf{a} + \frac{4}{7}\mathbf{b}$$

$$\overrightarrow{OH} = \frac{2}{7}\mathbf{a} + \frac{1}{7}\mathbf{b}$$

$$\overrightarrow{HF} = \frac{2}{7}\mathbf{a} + \frac{1}{7}\mathbf{b}$$

$$\overrightarrow{HG} = -\frac{1}{7}\mathbf{a} + \frac{3}{7}\mathbf{b}$$

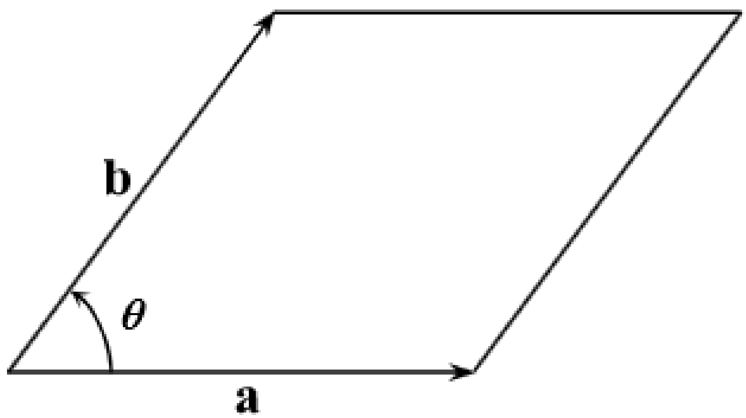
$$\text{Area } HFG = \frac{1}{2}\left|\left(\frac{2}{7}\mathbf{a} + \frac{1}{7}\mathbf{b}\right) \times \left(-\frac{1}{7}\mathbf{a} + \frac{3}{7}\mathbf{b}\right)\right|$$

$$\text{Area } HFG = \frac{1}{2}\left|\left(\frac{2}{7}\mathbf{a} + \frac{1}{7}\mathbf{b}\right) \times \left(-\frac{1}{7}\mathbf{a} + \frac{3}{7}\mathbf{b}\right)\right|$$

$\times$	$-\frac{1}{7}\mathbf{a}$	$+\frac{3}{7}\mathbf{b}$
$\frac{2}{7}\mathbf{a}$	$\mathbf{0}$	$\frac{6}{49}(\mathbf{a} \times \mathbf{b})$
$+\frac{1}{7}\mathbf{b}$	$-\frac{1}{49}(\mathbf{b} \times \mathbf{a}) = \frac{1}{49}(\mathbf{a} \times \mathbf{b})$	$\mathbf{0}$

$$\text{Area } HFG = \frac{1}{2}\left|\frac{7}{49}(\mathbf{a} \times \mathbf{b})\right|$$

$$= \frac{1}{7}\left(\frac{1}{2}|(\mathbf{a} \times \mathbf{b})|\right)$$



$$\text{Area} = ab\sin\theta = |\mathbf{a}\times\mathbf{b}|$$

The area of a parallelogram is $|\mathbf{a}\times\mathbf{b}|$