MEI CONFERENCE 2013

Keele University

Six Gems in FP1 and FP2

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Innovators in Mathematics Education

Keele University

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- Gem 6: A geometric view of Euler's Identity.

Matrices as functions

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This is very useful!!

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Can the function M'' be represented by a matrix?

Matrices as functions

$$\begin{aligned} x'' &= a'x' + c'y' \\ y'' &= b'x' + d'y' \end{aligned} \quad \text{and} \quad \begin{aligned} x' &= ax + cy \\ y' &= bx + dy \end{aligned}$$

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Now we define a way of combining (composing? multiplying?) two matrices:

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Proof. I'll prove (a) and part of (d) by way of example, and leave the proofs of the other parts to you.

Before starting, I should say that this proof is rather technical, but try to follow along as best you can. I'll use i, j, k, and l as subscripts.

Suppose that A is an $m \times n$ matrix, B is an $n \times p$ matrix, and C is a $p \times q$ matrix. I want to prove that $(AB)C = A(BC)$. I have to show that corresponding entries are equal, i.e.

$$((AB)C)_{il} = (A(BC))_{il}.$$

By definition of matrix multiplication,

$$((AB)C)_{il} = \sum_{k=1}^{p} (AB)_{ik} C_{kl} = \sum_{k=1}^{p} \left(\sum_{j=1}^{n} A_{ij} B_{jk} \right) C_{kl},$$

$$(A(BC))_{il} = \sum_{j=1}^{n} A_{ij} (BC)_{jl} = \sum_{j=1}^{n} A_{ij} \left(\sum_{k=1}^{p} B_{jk} C_{kl} \right).$$

If you stare at those two terrible double sums for a while, you can see that they involve the same A, B, and C terms, and they involve the same summations --- but in different orders. I'm allowed to convert one into the other by *interchanging the order of summation*, and using the distributive law:

$$\sum_{k=1}^{p} \left(\sum_{j=1}^{n} A_{ij} B_{jk} \right) C_{kl} = \sum_{k=1}^{p} \sum_{j=1}^{n} (A_{ij} B_{jk} C_{kl}) = \sum_{j=1}^{n} \sum_{k=1}^{p} (A_{ij} B_{jk} C_{kl}) = \sum_{j=1}^{n} A_{ij} \left(\sum_{k=1}^{p} B_{jk} C_{kl} \right).$$

Therefore, $((AB)C)_{il} = (A(BC))_{il}$, and so $(AB)C = A(BC)$. Wow!

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We have defined matrix multiplication so that it corresponds to function composition and function composition is obviously associative!

$$((f \circ g) \circ h)(x) = (f \circ (g \circ h))(x)$$

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$$((f \circ g) \circ h)(x) = (f \circ (g \circ h))(x)$$

Both sides mean "first apply h, then apply g, then apply f". I.e both sides reduce to $f(g(h(x)))$.

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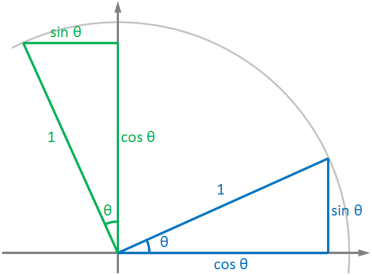

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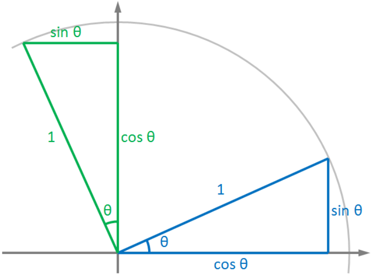

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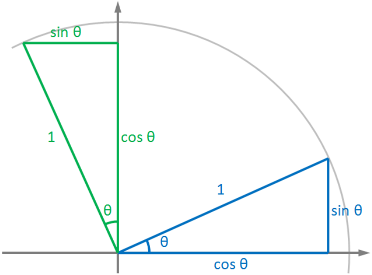

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So $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Gem 2: Compound angle formulae from matrix multiplication

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But LHS $= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix}$

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Hence

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
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$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)j$, so $M_{z_1+z_2} = \begin{pmatrix} x_1 + x_2 & -(y_1 + y_2) \\ y_1 + y_2 & x_1 + x_2 \end{pmatrix}$

$kz = kx + (ky)j$, so $M_{kz} = \begin{pmatrix} kx & -ky \\ ky & kx \end{pmatrix}$

$z_1 \times z_2 = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)j$, so

$$M_{z_1z_2} = \begin{pmatrix} x_1x_2 - y_1y_2 & -(x_1y_2 + x_2y_1) \\ x_1y_2 + x_2y_1 & x_1x_2 - y_1y_2 \end{pmatrix}$$

Gem 3: Complex numbers are isomorphic to (a subset of) matrices

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Multiplication as a rotation!

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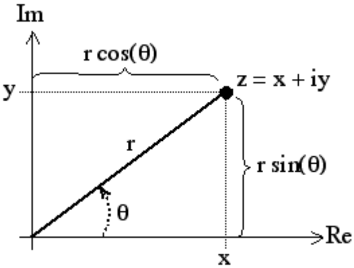
We can also represent a complex number as a point in the Argand plane.



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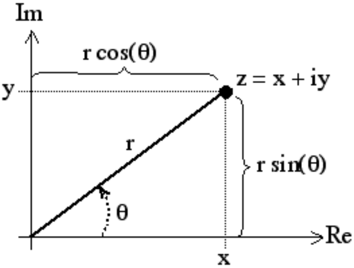

It follows that $z = r(\cos\theta + j\sin\theta)$, so $M_z = r\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$.

Gem 4: Multiplication by z is a rotation (and an enlargement)

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&= r_1 r_2 \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}
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Hence to multiply complex numbers in polar form you multiply their moduli and add their arguments.

Gem 5: Euler's identity

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Hence $z = re^{j\theta} = r\cos\theta + (r\sin\theta)j$. Setting $r = 1$, and $\theta = \pi$ gives

$$e^{j\pi} = -1.$$

Gem 5: Euler's Identity

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To see this, take a large value of n and take the binomial expansion without simplifying the coefficients.

$$\left(1+\frac{x}{100}\right)^{100} = 1 + 100\left(\frac{x}{100}\right) + \frac{100\times 99}{2!}\left(\frac{x^2}{100^2}\right) + \frac{100\times 99\times 98}{3!}\left(\frac{x^3}{100^3}\right) + \cdots$$

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$$e^{j\pi} = -1 \Rightarrow \lim_{n\to\infty} \left(1 + \frac{j\pi}{n}\right)^n = -1$$

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Extending this to complex numbers gives Euler's identity a geometric interpretation.

$$e^{j\pi} = -1 \Rightarrow \lim_{n\to\infty}\left(1+\frac{j\pi}{n}\right)^n = -1$$

So we should find that as n gets larger and larger $\left(1+\frac{j\pi}{n}\right)^n$ gets closer and closer to -1.

Gem 6: A geometric view of Euler's Identity

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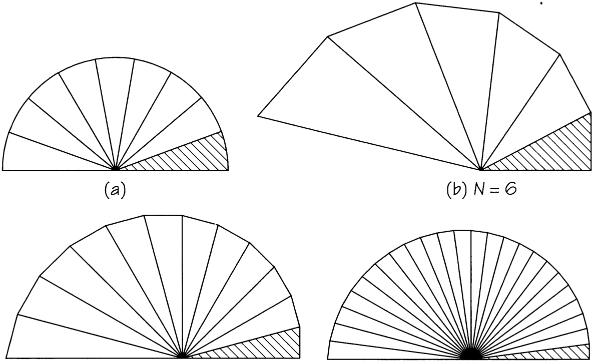

FIGURE 9.8 *Why* $e^{\pi i} = -1$.