MEI Conference 2013

Generating Functions

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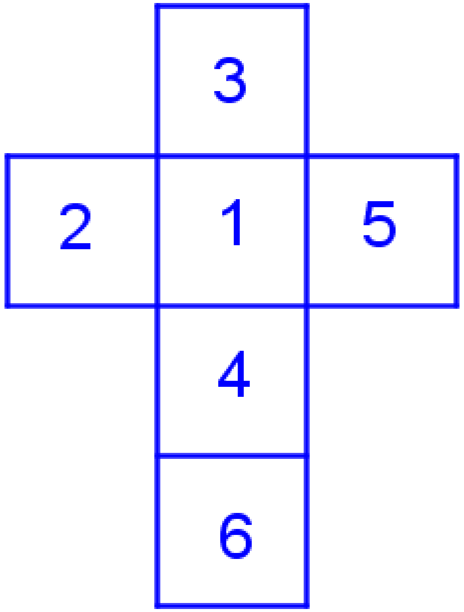
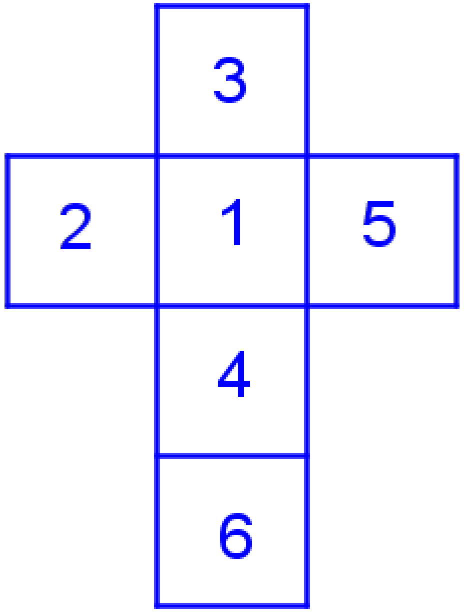
How would you go about finding two six sided dice, with a positive integer on each face but not the usual 1 to 6, which have the same distribution of sums as two normal dice?

There are 4 ways of writing 6 as the sum of distinct positive integers and also 4 ways of writing 6 as the sum of odd positive integers. Is this a coincidence?

In this session we'll introduce generating functions and see how they are helpful in answering these two questions.





	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

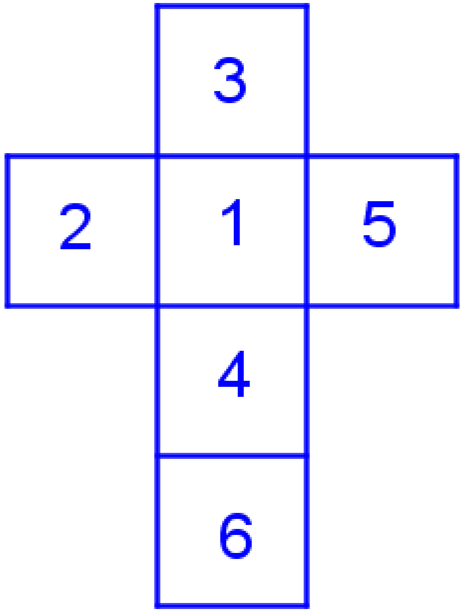
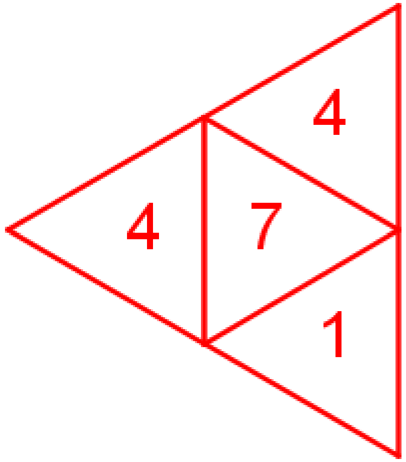
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

	0	1	2	3	4	5
2	2	3	4	5	6	7
3	3	4	5	6	7	8
4	4	5	6	7	8	9
5	5	6	7	8	9	10
6	6	7	8	9	10	11
7	7	8	9	10	11	12

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

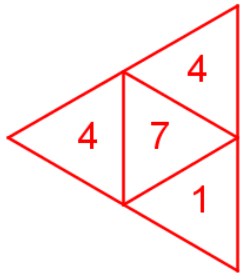
Two different 6-sided dice,
all positive integers which can be repeated,
same distribution of scores?

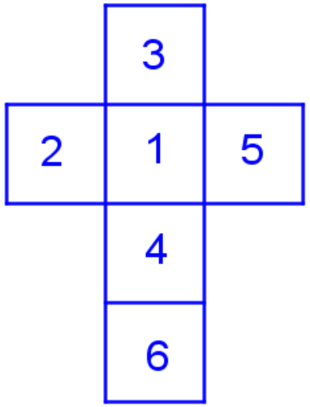
Generating functions

$f_1(x) = 1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6$	$f_2(x) = 1x^1 + 2x^4 + 1x^7$
	

$$f_1(x).f_2(x) = \left(1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6\right)\left(1x^1 + 2x^4 + 1x^7\right)$$

$$=$$





	1	4	4	7
1	2	5	5	8
2	3	6	6	9
3	4	7	7	10
4	5	8	8	11
5	6	9	9	12
6	7	10	10	13

$$\mathrm{f}_1(x).\mathrm{f}_2(x)=\left(1x^1+1x^2+1x^3+1x^4+1x^5+1x^6\right)\left(1x^1+2x^4+1x^7\right)$$
$$=1x^2+1x^3+1x^4+3x^5+3x^6+3x^7+3x^8+3x^9+3x^{10}+1x^{11}+1x^{12}+1x^{13}$$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$\mathrm{f}_1(x).\mathrm{f}_2(x) = \left(1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6\right)\left(1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6\right)$$

$$= 1x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + 1x^{12}$$

We want two other six-sided dice, with generating functions $g_1(x)$ and $g_2(x)$, which have the same distribution of totals from 2 to 12 as two normal dice.

$$g_1(x).g_2(x) = \left(1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6\right)^2$$

$$g_1(1) = 6 \quad \& \quad g_2(1) = 6$$

All coefficients in $g_1(x)$ and $g_2(x)$ are non-negative integers with no constant term. A non-zero constant term would imply that at least one side of the die is labelled 0, a possibility we'll exclude for now.

$$1x^1+1x^2+1x^3+1x^4+1x^5+1x^6=x\left(1+x^1+x^2+x^3+x^4+x^5\right)$$

$$=$$

$$1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6 = x\left(1 + x^1 + x^2 + x^3 + x^4 + x^5\right)$$

$$= x\left(1 + x + x^2\right)\left(1 + x^3\right)$$

$$= x\left(1 + x + x^2\right)\left(1 + x\right)\left(1 - x + x^2\right)$$

$$\mathrm{g}_1(x).\mathrm{g}_2(x) = \left(1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6\right)^2$$

$$= x^2\left(1 + x + x^2\right)^2\left(1 + x\right)^2\left(1 - x + x^2\right)^2$$

$$x \qquad\qquad x$$

$$1+x+x^2 \qquad\qquad 1+x+x^2$$

$$1+x \qquad\qquad 1+x$$

$$1-x+x^2 \qquad\qquad 1-x+x^2$$

$$\mathrm{g}_1\left(x\right)=$$

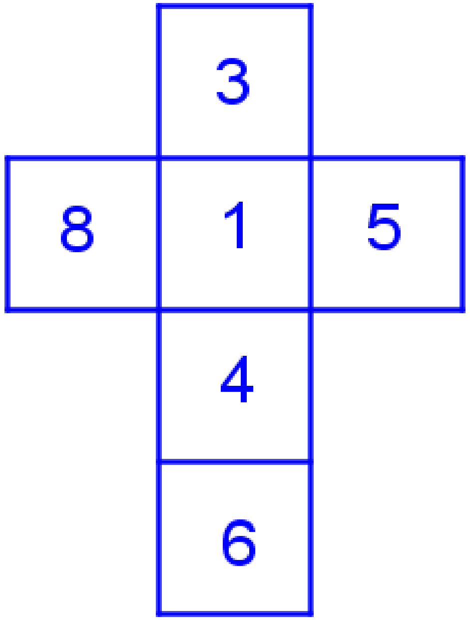
$$\mathrm{g}_2\left(x\right)=$$

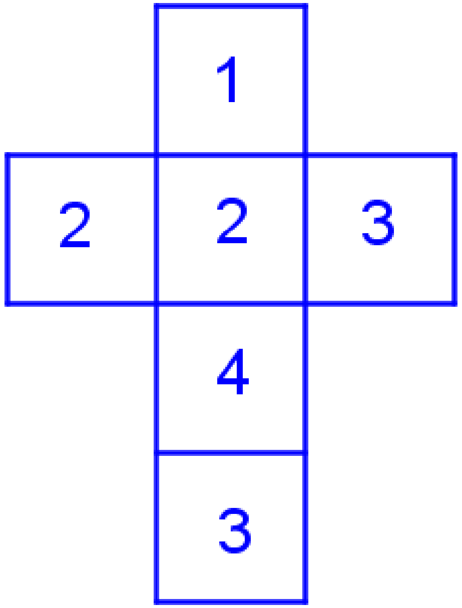
x	x
$1+x+x^2$	$1+x+x^2$
$1+x$	$1+x$
$1-x+x^2$	$1-x+x^2$

$$g_1(x)=(1+x)\left(1+x+x^2\right)^2$$

$$g_2(x)=x^2(1+x)\left(1-x+x^2\right)^2$$

No good because…





	1	3	4	5	6	8
1	2	4	5	6	7	9
2	3	5	6	7	8	10
2	3	5	6	7	8	10
3	4	6	7	8	9	11
3	4	6	7	8	9	11
4	5	7	8	9	10	12

What if we allowed non-cubical dice?

	1	2	2	3	3	3	4	4	5
1	2	3	3	4	4	4	5	5	6
4	5	6	6	7	7	7	8	8	9
4	5	6	6	7	7	7	8	8	9
7	8	9	9	10	10	10	11	11	12

Three dice?

Partitions

Partitions of 4	Partitions of 5
4	5
3+1	4+1
2+2	3+2
2+1+1	3+1+1
1+1+1+1	2+2+1
	2+1+1+1
	1+1+1+1+1

How many partitions of 6?....of 7?

n	$\mathrm{p}(n)$	
1	1	
2	2	
3	3	
4	5	
5	7	
6	11	
7	15	
8		
9		
10		
11		
12		

n	$\mathrm{p}(n)$	
1	1	
2	2	
3	3	
4	5	
5	7	
6	11	
7	15	
8	22	
9	30	
10	41	
11	55	
12	74	

n	$\mathrm{p}(n)$	
1	1	
2	2	
3	3	
4	5	
5	7	
6	11	
7	15	
8	22	
9	30	
10	41	
11	55	
12	74	
200	3,972,999,029,388	
243	133,978,259,344,888	

n	$\mathrm{p}(n)$	$e^{\pi\sqrt{2n/3}} \big/ 4n\sqrt{3}$
1	1	1.8
2	2	2.7
3	3	4.1
4	5	6.1
5	7	8.9
6	11	12.8
7	15	18.2
8	22	25.5
9	30	35.2
10	41	48.1
11	55	64.9
12	74	86.9
200	3,972,999,029,388	4.1×10^{12}
243	133,978,259,344,888	1.38×10^{14}

Leonhard Euler

1707 - 1783

D(n) is the number of ways of writing n as the sum of distinct positive integers.

For example, D(6) = 4:

6 & 5+1 & 4+2 & 3+2+1

n	1	2	3	4	5	6	7	8	9	10			
D(n)						4							

$O(n)$ is the number of ways of writing n as the sum of odd positive integers.

For example, $O(6) = 4$:

5+1 & 3+3 & 3+1+1+1 & 1+1+1+1+1+1

n	1	2	3	4	5	6	7	8	9	10			
$D(n)$	1	1	2	2	3	4	5	6	8	10			
$O(n)$						4							

Generating function

$$P(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$$

Generating function

$$Q(x)=\frac{1}{1-x}\cdot\frac{1}{1-x^{3}}\cdot\frac{1}{1-x^{5}}\cdot\frac{1}{1-x^{7}}\cdot\frac{1}{1-x^{9}}\cdots$$

$$\mathrm{P}(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$$
$$= 1 + \mathrm{D}(1)x + \mathrm{D}(2)x^2 + \mathrm{D}(3)x^3 + \mathrm{D}(4)x^4 + \ldots$$

$$\mathrm{Q}(x) = \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdot\frac{1}{1-x^9}\ldots$$
$$= (1 + x^1 + x^{1+1} + x^{1+1+1} + \ldots)(1 + x^3 + x^{3+3} + x^{3+3+3} + \ldots)\ldots$$
$$= 1 + \mathrm{O}(1)x + \mathrm{O}(2)x^2 + \mathrm{O}(3)x^3 + \mathrm{O}(4)x^4 + \ldots$$

“Read Euler, read Euler, he is the master of us all” - *Laplace*

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Generating Functions

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Sicherman Dice

Think about the function $f_1(x) = 1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6$. We'll call this the **generating function** for a normal die: the power indicates a possible outcome and the coefficient represents the number of ways of getting that outcome.

Therefore $h_1(x) = 1x^1 + 2x^4 + 1x^7$ is the generating function for a fair 4-sided die with sides numbered 1,4,4,7. Notice how $h_1(1) = 1 + 2 + 1 = 4$ gives the number of sides.

Similarly, $h_2(x) = 1x^1 + 2x^2 + 3x^3 + 2x^4 + 1x^5$ is the generating function for a fair 9-sided die (whatever a fair 9-sided die looks like!) with sides numbered 1,2,2,3,3,3,4,4,5. Notice how $h_2(1) = 1 + 2 + 3 + 2 + 1 = 9$ gives the number of sides.

Notice also that for two normal dice, where $f_1(x) = 1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6$ and $f_2(x) = 1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6$, then

$$\begin{aligned} f_1(x).f_2(x) &= \left(1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6\right)\left(1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6\right) \\ &= 1x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + 1x^{12} \end{aligned}$$

and the coefficient of x^n give the number of ways of getting a total of n when the two dice are rolled.

Now if we want two other six-sided dice, with generating functions $g_1(x)$ and $g_2(x)$, which have the same probabilities for each total from 2 to 12 as two normal dice, then $g_1(x)$ and $g_2(x)$ must have the following three properties:

1. $g_1(x).g_2(x) = \left(1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6\right)^2$ so that there is one way of getting 2, two ways of getting 3, and so on;
2. $g_1(1) = 6$ and $g_2(1) = 6$ to ensure the dice have six sides;
3. All coefficients are non-negative integers with no constant term. A non-zero constant term would imply that at least one side of the die is labelled 0, a possibility we'll exclude for now.

To find possible generating functions with these three properties, first factorise $1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6$:

$$\begin{aligned} x\left(1 + x^1 + x^2 + x^3 + x^4 + x^5\right) &= x\left(1 + x + x^2\right)\left(1 + x^3\right) \\ &= x\left(1 + x + x^2\right)\left(1 + x\right)\left(1 - x + x^2\right) \end{aligned}$$

Therefore $g_1(x).g_2(x) = \left(1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6\right)^2$

$$= x^2\left(1 + x + x^2\right)^2\left(1 + x\right)^2\left(1 - x + x^2\right)^2$$

Property 2 states $g_1(1)=6$ and $g_2(1)=6$. This means that both of the generating functions must have both $(1+x+x^2)$ and $(1+x)$ as factors.
They must also each have a factor of x, otherwise on multiplying out you would find that the constant term is equal to 1 and so there would be one side labelled 0.

So there are two possibilities:

1. $g_1(x)=x(1+x+x^2)(1+x)(1-x+x^2)$, $g_2(x)=x(1+x+x^2)(1+x)(1-x+x^2)$
2. $g_1(x)=x(1+x+x^2)(1+x)(1-x+x^2)^2$, $g_2(x)=x(1+x+x^2)(1+x)$

To find the values on the faces of the corresponding dice we need to multiply out these expressions. This gives:

1. $g_1(x)=1x^1+1x^2+1x^3+1x^4+1x^5+1x^6$, $g_2(x)=1x^1+1x^2+1x^3+1x^4+1x^5+1x^6$
2. $g_1(x)=1x^1+1x^3+1x^4+1x^5+1x^6+1x^8$, $g_2(x)=1x^1+2x^2+2x^3+1x^4$

The first of these gives the usual dice with sides labelled 1,2,3,4,5,6, whereas the second pair has sides labelled 1,3,4,5,6,8 and 1,2,2,3,3,4. These are the Sicherman Dice.

Under property 3, we excluded the possibility of labelling any sides 0. If we allowed this then there would be three extra possibilities in addition to the two given:

3. $g_1(x)=(1+x+x^2)(1+x)x^2(1-x+x^2)^2$, $g_2(x)=(1+x+x^2)(1+x)$
4. $g_1(x)=(1+x+x^2)(1+x)x^2(1-x+x^2)$, $g_2(x)=(1+x+x^2)(1+x)(1-x+x^2)$
5. $g_1(x)=(1+x+x^2)(1+x)x^2$, $g_2(x)=(1+x+x^2)(1+x)(1-x+x^2)^2$

Why do these turn out to be relatively uninteresting?

Returning to the generating functions $h_1(x)=1x^1+2x^4+1x^7$ and $h_2(x)=1x^1+2x^2+3x^3+2x^4+1x^5$, you will find that these, too, have the same distribution of totals as two standard dice. Can you find any more?

	1	2	2	3	3	3	4	4	5
1	2	3	3	4	4	4	5	5	6
4	5	6	6	7	7	7	8	8	9
4	5	6	6	7	7	7	8	8	9
7	8	9	9	10	10	10	11	11	12

Partitions

Let D(*n*) represent the number of ways of writing *n* as the sum of distinct positive integers. For example, $\mathrm{D}(6)=4$ from the partitions $6\ \&\ 5+1\ \&\ 4+2\ \&\ 3+2+1$.

Let O(*n*) represent the number of ways of writing *n* as the sum of odd positive integers. For example, $\mathrm{O}(6)=4$ from the partitions $5+1\ \&\ 3+3\ \&\ 3+1+1+1\ \&\ 1+1+1+1+1+1$.

For any value of n you will find that $\mathrm{D}(n)=\mathrm{O}(n)$. Here is Euler's proof of this result.

First, consider the function $\mathrm{P}(x)=(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)...$
In the expansion, x^6 arises in exactly 4 ways; this 4 corresponding to $\mathrm{D}(6)$:

$$x^1.x^2.x^3.1.1.1.1... \quad + \quad x^1.1.1.1.x^5.1.1... \quad + \quad 1.x^2.1.x^4.1.1.1.... \quad + \quad 1.1.1.1.1.x^6.1....$$

It can be seen that $\mathrm{P}(x)=1+\mathrm{D}(1)x+\mathrm{D}(2)x^2+\mathrm{D}(3)x^3+\mathrm{D}(4)x^4+...$

Now consider the function $\mathrm{Q}(x)=\dfrac{1}{1-x}.\dfrac{1}{1-x^3}.\dfrac{1}{1-x^5}.\dfrac{1}{1-x^7}.\dfrac{1}{1-x^9}...$

Thinking about infinite geometric series, or the binomial theorem, for $-1<x<1$ this is the same as

$$\mathrm{Q}(x)=\left(1+x^1+x^{1+1}+x^{1+1+1}+...\right)\left(1+x^3+x^{3+3}+x^{3+3+3}+...\right)\left(1+x^5+x^{5+5}+x^{5+5+5}+...\right)...$$

In the expansion, x^6 arises in exactly 4 ways; this 4 corresponding to $\mathrm{O}(6)$:

$$x^{1+1+1+1+1+1}.1.1.1... \quad + \quad x^{1+1+1}.x^3.1.1... \quad + \quad x^1.1.x^5.1.... \quad + \quad 1.x^{3+3}.1.1...$$

It can be seen that $\mathrm{Q}(x)=1+\mathrm{O}(1)x+\mathrm{O}(2)x^2+\mathrm{O}(3)x^3+\mathrm{O}(4)x^4+...$

From the outset, Euler realised that $\mathrm{P}(x)\equiv\mathrm{Q}(x)$:

$$\begin{aligned}\mathrm{P}(x)&=(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)...\\&=(1+x)\times1\times(1+x^2)\times1\times(1+x^3)\times1\times(1+x^4)\times1\times(1+x^5)...\\&=\cancel{(1+x)}\,\frac{\cancel{(1-x)}}{(1-x)}\,\cancel{(1+x^2)}\,\frac{\cancel{(1-x^2)}}{\cancel{(1-x^2)}}(1+x^3)\frac{(1-x^3)}{(1-x^3)}(1+x^4)\frac{(1-x^4)}{\cancel{(1-x^4)}}...=\mathrm{Q}(x)\end{aligned}$$

Therefore $\mathrm{D}(n)=\mathrm{O}(n)$ for all values of n.