

MEI STRUCTURED MATHEMATICS

Marking C3 Coursework

**10 tips to ensure that the right mark is
awarded**

Marking C3 Coursework

C3 coursework is very prescriptive. Providing assessors follow the criteria carefully there is no problem with the assessment.

However, there are difficulties for the External moderator.

- Errors are made in the marking
- Work is not checked but assumed to be correct.
- Credit is given for work that is not evident.

It is not a question of “what is a good piece of coursework?” but “how can I ensure that I give an appropriate mark?”

1 Terminology

This task is all about solving equations. Therefore, candidates should write equations.

Persistent errors should be penalised in domain 5.

Examples which should be penalised:

I am going to solve the equation $x^3 - 4x - 1$.

I am going to solve the equation $y = x^3 - 4x - 1$.

I am going to solve the equation $f(x) = x^3 - 4x - 1$.

Correct terminology: I am going to solve the equation $x^3 - 4x - 1 = 0$.

Or I am going to solve the equation $f(x) = 0$ where $f(x) = x^3 - 4x - 1$.

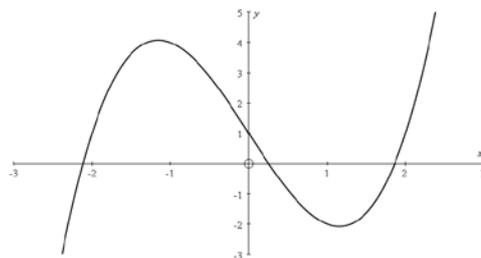
2 Illustrations

All three methods require a graph and an illustration for both success and failure. A graph of the function is not an illustration of the method.

Example

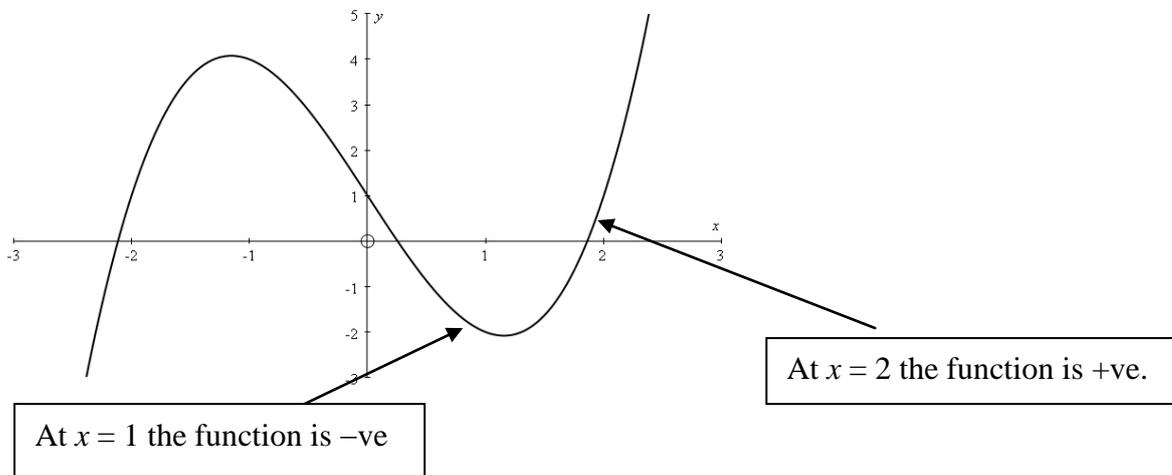
I am going to solve the equation $x^3 - 4x - 1 = 0$

Here is a graph of $y = x^3 - 4x - 1$.

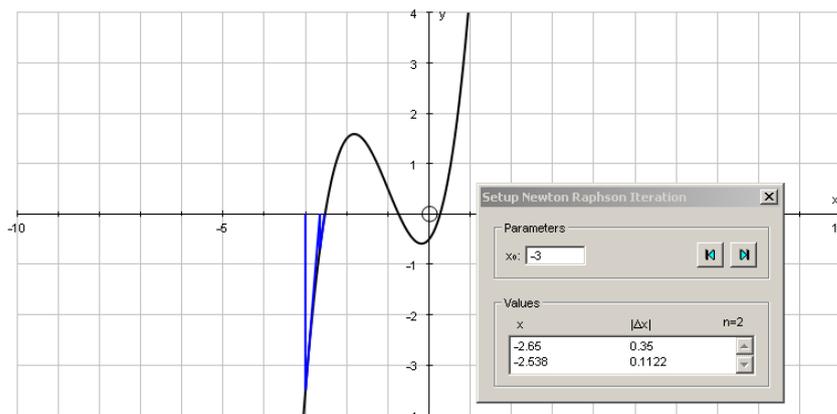


You can see from the graph that there is a root of the equation in the range $[1,2]$ which I shall find.
This is not enough – only the graph has been drawn.
 For the change of sign method there needs to be some annotation or some “zoom in” graphs.

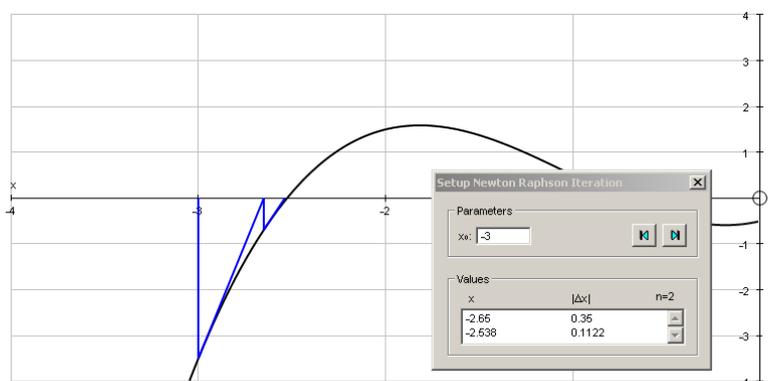
Example



For the Newton-Raphson method there needs to be two clear tangents showing convergence.



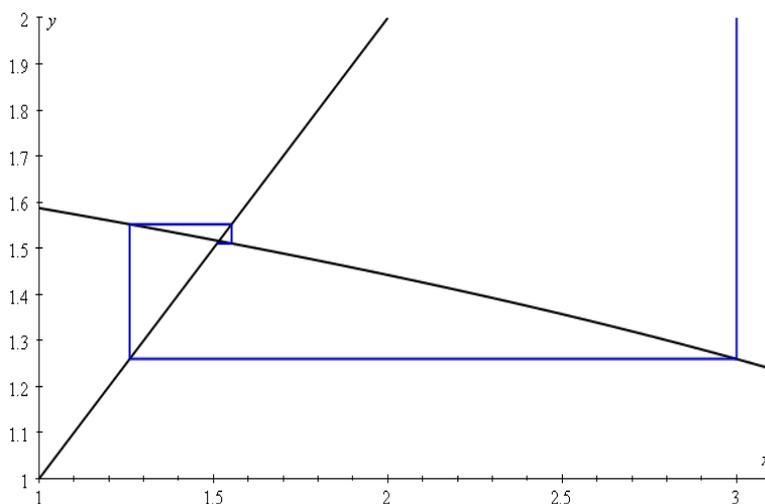
This is not clear enough!



This is better.

For the rearrangement method a cobweb or staircase diagram should be clear.

E.g. Consider the solution of the equation $x^3 + x - 5 = 0$ which rearranges to $x = \sqrt[3]{5-x}$.



Note that general theory is not necessary or credit-worthy. It is not necessary to derive the Newton-Raphson formula and certainly not acceptable to credit such work as an illustration of the method. Each illustration should match the iterates used.

3 Iterates

Every example of success and failure needs some iterates. Often these are missing yet credit is given for general theory.

The iterates must match the illustrations.

For example, the diagrams above illustrating the Newton-Raphson Method were of the function $y = x^3 + 3x^2 + x - 0.5$. This is the function that should be used when finding roots to demonstrate success.

Excel screen shots are acceptable but the algebra must be shown.

E.g. I am going to solve the equation $x^3 + 3x^2 + x - 0.5 = 0$.

The graphs above illustrate the method.

$$f(x) = x^3 + 3x^2 + x - 0.5$$

$$f'(x) = 3x^2 + 6x + 1$$

The Newton-Raphson formula gives

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x)}{f'(x)} = x_n - \frac{x^3 + 3x^2 + x - 0.5}{3x^2 + 6x + 1} \\ &= \frac{2x^3 + 3x^2 + 0.5}{3x^2 + 6x + 1} \end{aligned}$$

The roots are found using an Excel spreadsheet

1	0.55
0.55	0.334181
0.334181	0.272347
0.272347	0.267073
0.267073	0.267035

-1	-0.75
-0.75	-0.74138
-0.74138	-0.74135
-0.74135	-0.74135
-0.74135	-0.74135

-3	-2.65
-2.65	-2.53778
-2.53778	-2.52582
-2.52582	-2.52569
-2.52569	-2.52569

(N.B. Error bounds need to be established for one of these roots.)

4 Accuracy

The criteria state that the roots found using the change of sign and rearrangement method should be accurate to 0.5×10^{-3} and the Newton-Raphson method to 5 significant figures.

Users of “Autograph” need to learn how to produce roots to more than the default 4 significant figures!

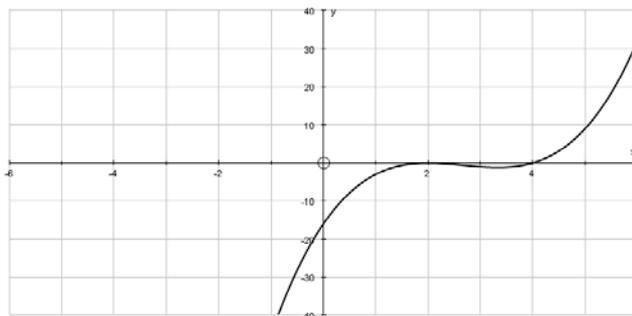
5 Failure of methods

Each method should have a demonstration of how it might fail with an equation that is not trivial.

Typical problems with trivial equations.

- (i) In the change of sign method the root is found within the body of the table.

E.g. $y = x^3 - 8x^2 + 20x - 16$



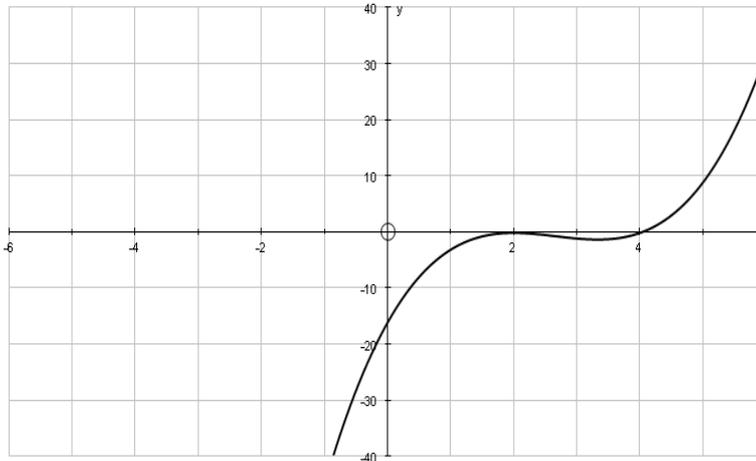
1	-3
1.1	-2.349
1.2	-1.792
1.3	-1.323
1.4	-0.936
1.5	-0.625
1.6	-0.384
1.7	-0.207
1.8	-0.088
1.9	-0.021
2	0
2.1	-0.019
2.2	-0.072
2.3	-0.153

“You can see that there is no change in sign so the method has failed.”

But the root has been found so the method has not failed!

(ii) In the change of sign method there is no root.

E.g. $y = x^3 - 8x^2 + 20x - 16.2$



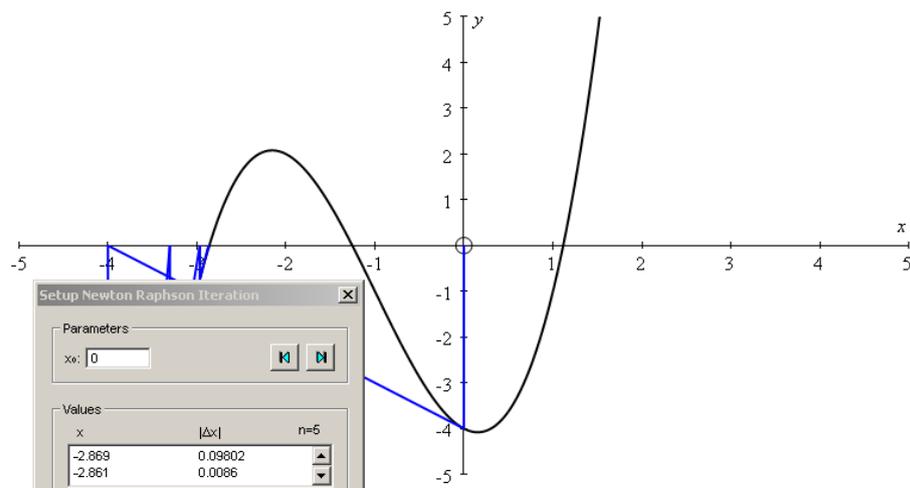
1	-3.2
1.1	-2.549
1.2	-1.992
1.3	-1.523
1.4	-1.136
1.5	-0.825
1.6	-0.584
1.7	-0.407
1.8	-0.288
1.9	-0.221
2	-0.2
2.1	-0.219
2.2	-0.272
2.3	-0.353

“You can see that there is no change in sign so the method has failed to find the root.”

The method has in fact worked because it has shown that there is no root where there is no root.

(iii) In the Newton-Raphson method the initial value is not close enough.

If the root to be found lies in the unit interval $[a, a + 1]$ then $x_0 = a$ or $x_0 = a + 1$ should be used. With the right choice of equation the method can be shown to converge to one of the roots not within the unit interval even when starting with one of the end points.



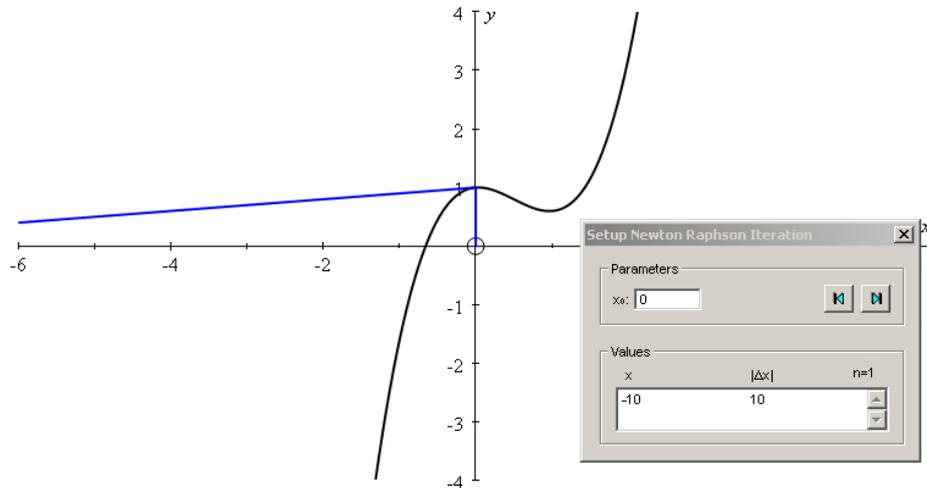
In this example there is a root in the interval $[1, 2]$. Starting with $x_0 = 0$ will result in convergence to a different root.

- (iv) In the Newton-Raphson Method an assumption is made that if x_2 is manifestly further way than x_1 then the method has failed.

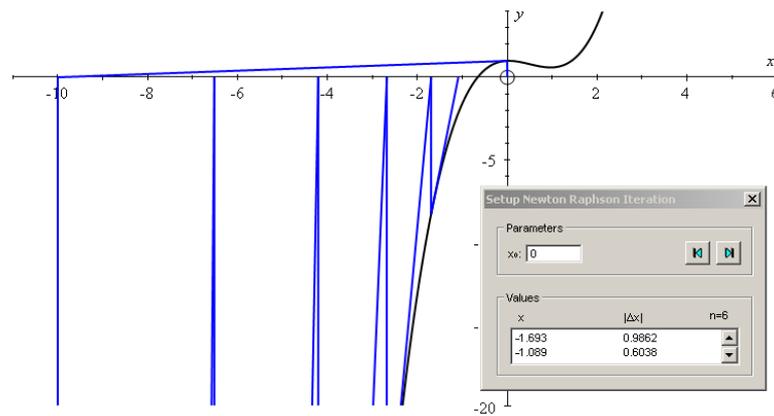
E.g. for the curve $y = x^3 - 1.5x^2 + 0.1x + 1$.

There is a root of the equation $y = 0$ in interval $[-1,0]$.

So start with $x=0$. You can see from the graph that the method is diverging.



It should be clear that Newton-Raphson always works! Enough iterations will indicate convergence.



Other trivial equations that are sometimes used and given credit particularly in the change of sign method.

$$\frac{1}{x} = 0, \quad \frac{1}{x-3} = 0, \quad (x-1.1)(x-1.2) = 0$$

6 Error bounds.

Error bounds need to be stated and justified in the first two methods. We do not demand this for the rearrangement method.

For the change of sign, the error bounds are built in to the method but nonetheless need to be stated. For the Newton-Raphson method the bounds need to be justified by change of sign. It is not good enough to state bounds on the basis of a table of values unless it can be shown that in this instance the method is working as a second order method. This is clearly outside the syllabus and not expected.

E.g. If, to 5 sf, $x_1 = 1.2344$, $x_2 = 1.2344$ then it is not true to say that, to 5sf, $x = 1.2344$ because it might be that to 6sf, $x_1 = 1.23443$, $x_2 = 1.23444$, $x_3 = 1.23445$. In this example you could not even justify, $x = 1.2345$.

7 Roots

An equation has one solution. The solution of $f(x) = 0$ is the set of values of x that satisfy the equation. One of these values is called a root

In the first and third domain one root is required. In the Newton-Raphson method all roots are required. If all three roots of a cubic equation are not found then the second mark should not be awarded. If there is only one root to the equation chosen then the second mark cannot be awarded.

In the first and third domain, roots are required to be found to 3 decimal places. In the Newton-Raphson method the level of accuracy is 5 sf. If this is not done then a penalty should be applied. Candidates using “Autograph” should be taught how to increase the number of significant figures from the default number of 4.

The root found in the change of sign method should be given and not left as a bound.

E.g. $f(1.234) > 0$, $f(1.235) < 0$ should not be left as “there is a root in the interval $[1.234, 1.235]$ ”.

The mark for the error bounds can be given but not the root.

The root is $x = 1.2345 \pm 0.0005$ is the correct conclusion.

8 The $g'(x)$ problem

When $f(x) = 0$ is rearranged to give $x = g(x)$ the criterion for convergence is $-1 < g'(x) < 1$.

There are two ways of showing this.

- (i) A clear discussion showing the curve $y = g(x)$ and the line $y = x$ intersecting with the implications of how they cross and the gradient of the line.
- (ii) Calculate $g'(x)$ near to the root and show that it satisfies the criterion $-1 < g'(x) < 1$.

9 Comparison of methods

It is expected that one equation used for success in one method now be used to find the same root to the same level of accuracy. For a meaningful comparison of speed of convergence you should also start with the same initial value.

E.g. Rearrangement method: $x_1 = 1, \dots, x_{30} = 2.3412$; convergence has occurred in 30 iterations
Newton-Raphson Method, $x_1 = 2, \dots, x_6 = 2.3412$; convergence has occurred in 6 iterations
and so is much faster. *This is not a valid comparison as the starting points are different.*

10 Ease of use

This is in two parts; often they are confused, but credit is given.

Candidates are expected to comment on the merits of each method. This could include how well they worked, what the significance of failure possibilities etc.

Candidates should then comment on the ease or difficulty of use with the available software and hardware. The conclusions of two candidates may be entirely different dependent on what resources they had available.

For instance: “The change of sign method is much easier if a spreadsheet is used” is a general, and correct, statement. “I found that finding the root by the change of sign method was much easier than it might have been because I used an Excel spreadsheet and was able to “copy down” values for different intervals and to copy them across into the report” tells me something about what the candidate used and what he/she thought about it.

Errors in marking where a penalty of ½ mark should be applied and often is not.

- Terminology
- Illustrations (possibly $3 \times \frac{1}{2}$ here!)
- Change of sign root not given.
- Change of sign failure not satisfactory
- Not enough significant figures in the Newton-Raphson method.
- Only one root in the equation used in the Newton-Raphson method (a penalty of 1 mark here).
- Error bounds in the Newton-Raphson method are stated but not justified, or are not even stated.
- Failure of the Newton-Raphson method: not a failure deliberately and unusual choice of starting value deliberately chosen because it is known to be a stationary value.
- Comparison: The same root is not found to the same level of accuracy from the same initial value.
- The actual software used by the candidate is not specifically mentioned.

Each of these errors typically should incur a penalty of ½ mark. Failure to penalise in 4 areas means a reduction of 2 which is outside the tolerance allowed and will trigger a scaling of marks.