MEI
Innovators in Mathematics Education

Mathematics in Education and Industry

50 years at the forefront of Mathematics Education

Ideas for a Further Mathematics classroom

FIVE USES OF AN EIGENVECTOR

Phil Chaffé 2013

What do we teach students when we look at eigenvectors and eigenvalues in FP2?

An *eigenvector* of a square matrix **M** is a non-zero vector **s** such that $\mathbf{Ms} = \lambda\mathbf{s}$; the scalar λ is the corresponding *eigenvalue.*

The characteristic equation of **M** is $|\mathbf{M} - \lambda\mathbf{I}| = 0$

If **S** is the matrix formed of the eigenvectors of **M** and **Λ** is the diagonal matrix formed of the corresponding eigenvalues then $\mathbf{MS} = \mathbf{S\Lambda}$, $\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{MS}$, $\mathbf{M}^n = \mathbf{S\Lambda}^n\mathbf{S}^{-1}$

Some helpful diagrams

Invariant lines

Transforming a circle

What can eigenvectors and eigenvalues be used for?

Schrödinger equation
Molecular orbitals
Geology and glaciology
Principal components analysis
Vibration analysis
Eigenfaces
Tensor of moment of inertia
Stress tensor
Eigenvalues of a graph
Basic reproduction number

…thanks Wikipedia

Suitable applications for A level FM students…

- Use mathematics that they "know"

- Examples from each of
  - Pure Mathematics
  - Mechanics
  - Statistics
  - Decision Mathematics

- Five of them (originally 10!)

Find eigenvalues and eigenvectors of these matrices

$$\begin{pmatrix} 2 & -9 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}$$

Pure Mathematics

Find the solutions to this pair of differential equations

$\frac{dx}{dt} = 2x$, $\frac{dy}{dt} = -3y$ given $x = 1$ and $y = 9$ at $t = 0$

The original equation could have been written in the form:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$$

Where $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \dot{\mathbf{X}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

Coupled differential equations

$$\dot{x} = 2x - 9y$$
$$\dot{y} = -x + 2y$$

$$t = 0, x = 1, y = 0$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & -9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{\mathbf{X}} = \mathbf{AX}$$

Introduce a new vector $\mathbf{Y} = \begin{pmatrix} r \\ s \end{pmatrix}$ where $\mathbf{X} = \mathbf{SY}$ and $\mathbf{S}$ is the modal matrix of $\mathbf{A}$

$$\mathbf{X} = \mathbf{SY} \Rightarrow \dot{\mathbf{X}} = \mathbf{S}\dot{\mathbf{Y}}$$

so $\dot{\mathbf{X}} = \mathbf{AX}$ becomes $\mathbf{S}\dot{\mathbf{Y}} = \mathbf{AX}$

$$\mathbf{S}\dot{\mathbf{Y}} = \mathbf{A}(\mathbf{SY})$$
$$\dot{\mathbf{Y}} = \mathbf{S}^{-1}\mathbf{ASY}$$
$$\dot{\mathbf{Y}} = \left(\mathbf{S}^{-1}\mathbf{AS}\right)\mathbf{Y}$$

$\mathbf{S}^{-1}\mathbf{AS}$ is the diagonal matrix $\mathbf{\Lambda}$

$$\begin{pmatrix} \dot{r} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}$$

$$\dot{r} = ar$$
$$\dot{s} = bs$$

$$\dot{x} = 2x - 9y$$
$$\dot{y} = -x + 2y$$

$$t = 0, x = 1, y = 0$$

$$\mathbf{A} = \begin{pmatrix} 2 & -9 \\ -1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2 - \lambda & -9 \\ -1 & 2 - \lambda \end{vmatrix} = 0$$

$$(2 - \lambda)^2 - 9 = 0$$

$$\lambda_1 = -1, \lambda_2 = 5$$

$$\lambda_1 = -1, \mathbf{s}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 5, \mathbf{s}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix} \qquad \boldsymbol{\Lambda} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} \dot{r} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}$$

$$\dot{r} = -r$$
$$\dot{s} = 5s$$

$$\mathbf{X} = \mathbf{S}\mathbf{Y}$$

$$r = Ae^{-t}$$
$$s = Be^{5t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} Ae^{-t} \\ Be^{5t} \end{pmatrix}$$

$$x = 3Ae^{-t} - 3Be^{5t}$$
$$y = Ae^{-t} + Be^{5t}$$

$$x = 3Ae^{-t} - 3Be^{5t} \qquad 3A - 3B = 1$$

$$y = Ae^{-t} + Be^{5t} \qquad A + B = 0$$

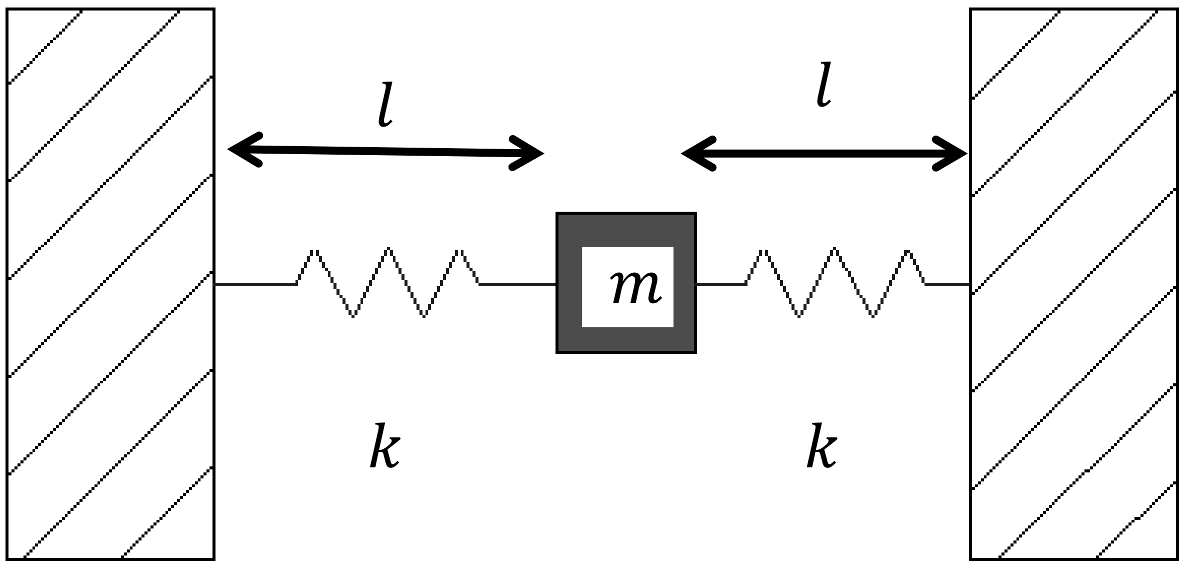
$$t = 0, x = 1, y = 0$$

$$A = \frac{1}{6}, B = -\frac{1}{6}$$

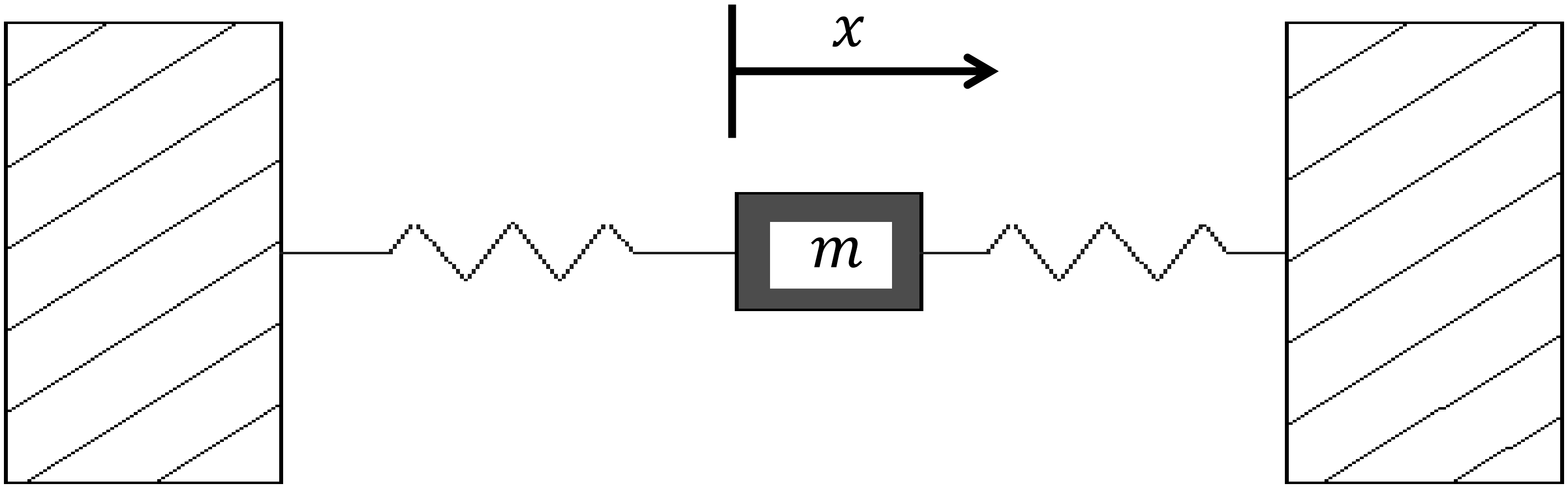
$$x = \frac{1}{2}e^{-t} + \frac{1}{2}e^{5t}$$

$$y = \frac{1}{6}e^{-t} - \frac{1}{6}e^{5t}$$

Mechanics



$$T = kx$$

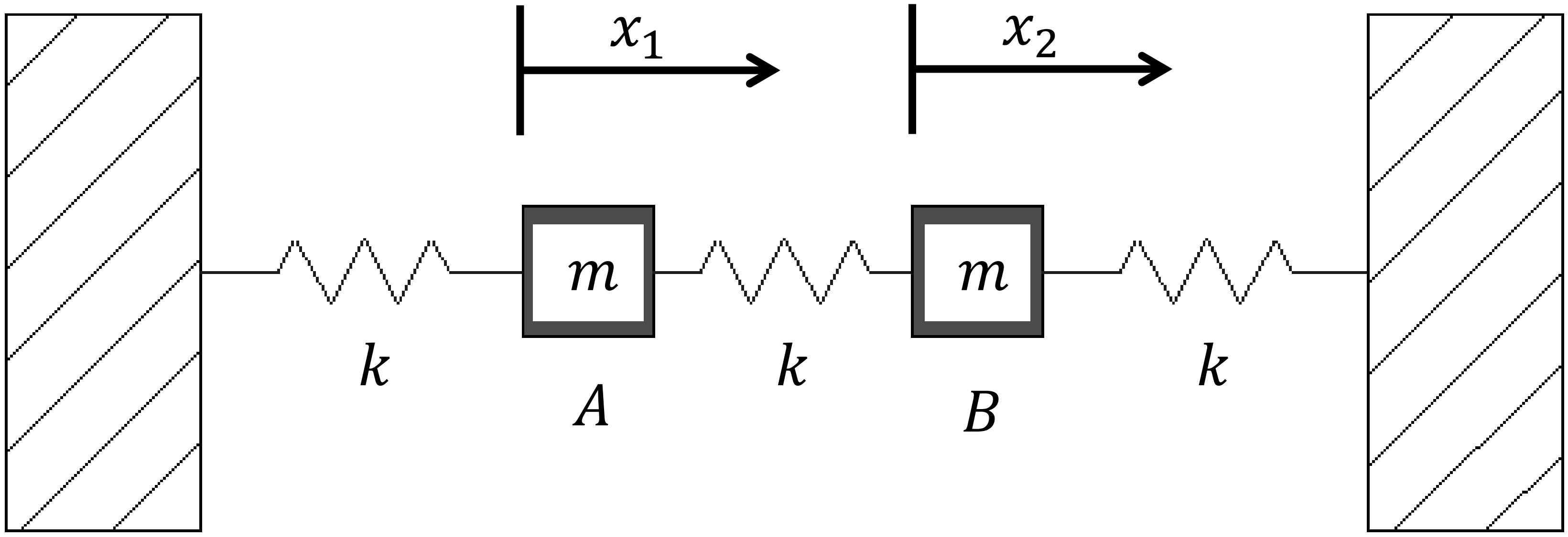


$$\ddot{x} + \omega^2 x = 0$$

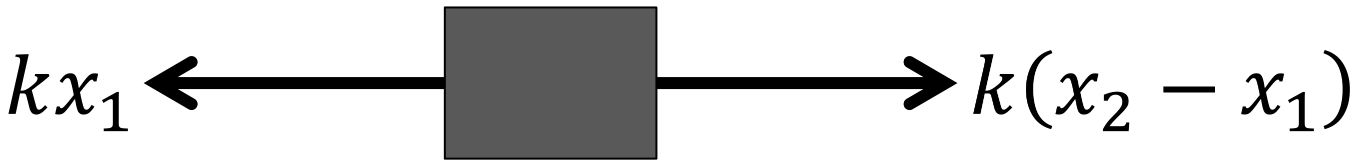
$$\omega^2 = {}^{2k}/_{m}$$

$$x = a\sin(\omega t + \varepsilon)$$

So what happens with ….



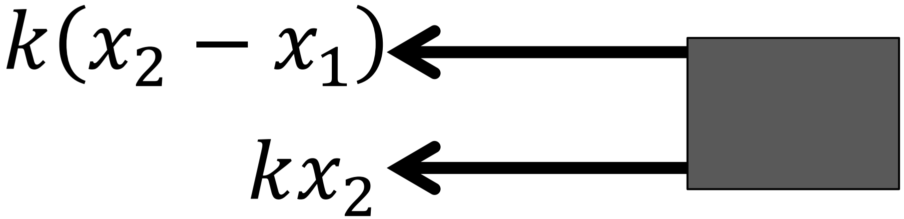
A



$$m\ddot{x}_1 = k(x_2 - x_1) - kx_1$$

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2$$

B



$$m\ddot{x}_2 = -k(x_2 - x_1) - kx_2$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2k/m & k/m \\ k/m & -2k/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\ddot{\boldsymbol{x}} = \mathbf{M}\boldsymbol{x}$$

$$\ddot{\boldsymbol{x}} - \mathbf{M}\boldsymbol{x} = 0$$

Assume the form of the solution is

$$\boldsymbol{x} = \mathbf{v_1} e^{j\omega t} + \mathbf{v_2} e^{-j\omega t}$$

$$\dot{\boldsymbol{x}} = j\omega \mathbf{v_1} \boldsymbol{e^{j\omega t}} - j\omega \mathbf{v_2} e^{-j\omega t}$$

$$\ddot{\boldsymbol{x}} = j^2\omega^2 \mathbf{v_1} \boldsymbol{e^{j\omega t}} + j^2\omega^2 \mathbf{v_2} e^{-j\omega t}$$

$$\ddot{\boldsymbol{x}} = -\omega^2 \mathbf{v_1} \boldsymbol{e^{j\omega t}} - \omega^2 \mathbf{v_2} e^{-j\omega t}$$

$$\ddot{\boldsymbol{x}} = -\omega^2 \boldsymbol{x}$$

$$\ddot{\boldsymbol{x}} - \mathbf{M}\boldsymbol{x} = 0$$

$$-\omega^2\boldsymbol{x} - \mathbf{M}\boldsymbol{x} = 0$$

$$\mathbf{M}\boldsymbol{x} = -\omega^2\boldsymbol{x}$$

$$\mathbf{M}\boldsymbol{x} = \lambda\boldsymbol{x}$$

$$|\mathbf{M} - \lambda\mathbf{I}| = \mathbf{0}$$

Demonstrations

Two body 3 spring system

Why is this important

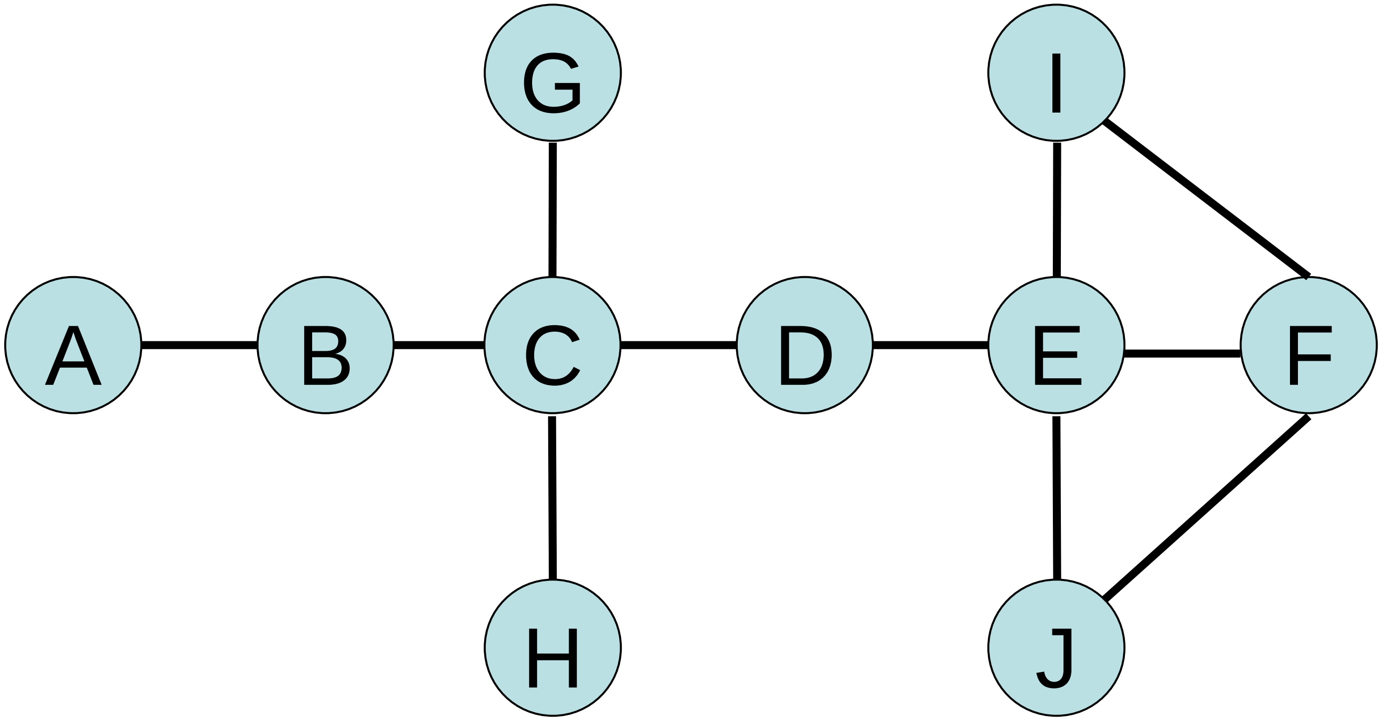
Tacoma bridge

Decision Mathematics

Centrality:

- The relative importance of a node within a graph.

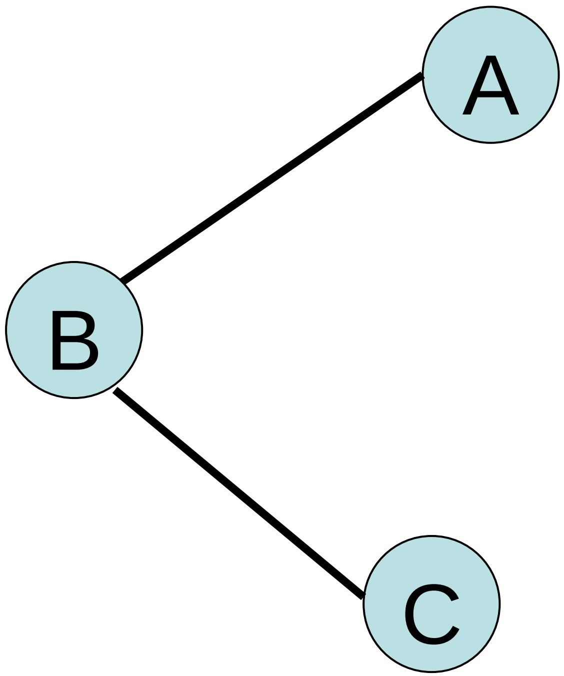
- Various measures of this
  - degree centrality
  - Gould's Index (eigenvector centrality)



Degree centrality: $C(v) = \frac{\deg(v)}{n-1}$

For B: $C(B) = \frac{2}{9}$

For D: $C(D) = \frac{2}{9}$



Adjacency matrix

$$\mathbf{M} = \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \begin{matrix} A \\ B \\ C \end{matrix}$$

Use the normalised eigenvector from the principal eigenvalue (i.e. the largest eigenvalue)

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 - 1) - 1(-\lambda) = 0$$

$$\lambda(\lambda^2 - 2) = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = +\sqrt{2}$$

$$\lambda_3 = -\sqrt{2}$$

$$\lambda_2 = +\sqrt{2}$$

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x = \frac{1}{\sqrt{2}}y \quad z = \frac{1}{\sqrt{2}}y$$

$$\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\mathbf{v_2} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

B is clearly the "most important" node

Quadratic Forms

Augustin Louis Cauchy

$$ax^2 + 2bxy + cy^2 = d$$

If $b \neq 0$, the curve is a conic rotated about O.

Find new coordinate axes for the graph so that the equation with respect to these new axes is of the form $Au^2 + Bv^2 = D$.

Example

$$8x^2 + 4xy + 5y^2 = 1$$

$$\boldsymbol{x}^T\mathbf{A}\boldsymbol{x} = 1$$

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

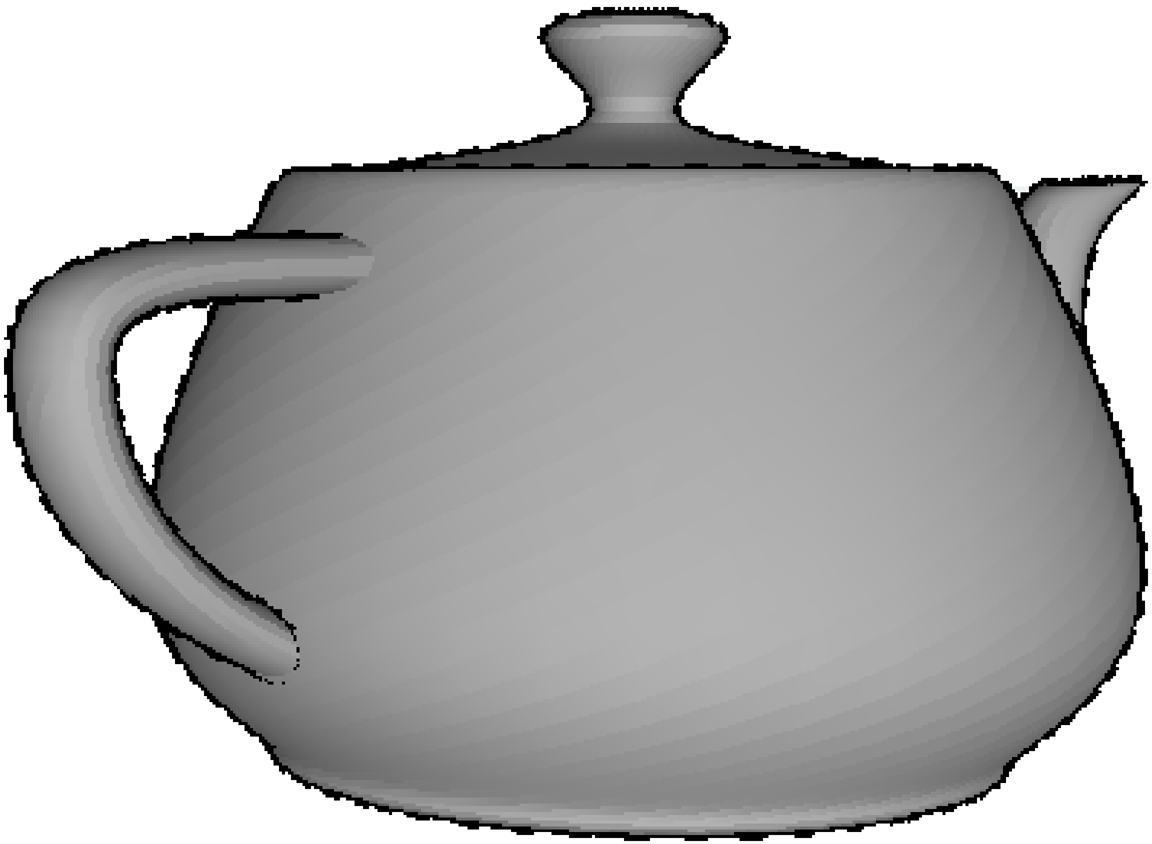
$$\begin{vmatrix} 8-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 13\lambda + 36 = 0$$

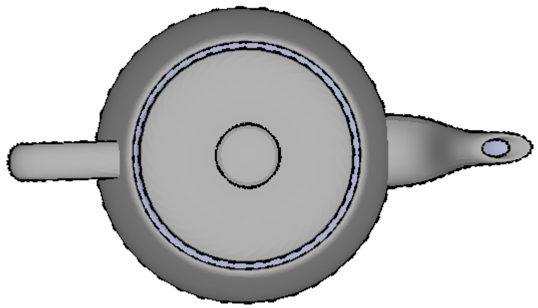
$$\lambda_1 = 4 \qquad \mathbf{s}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = 9 \qquad \mathbf{s}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

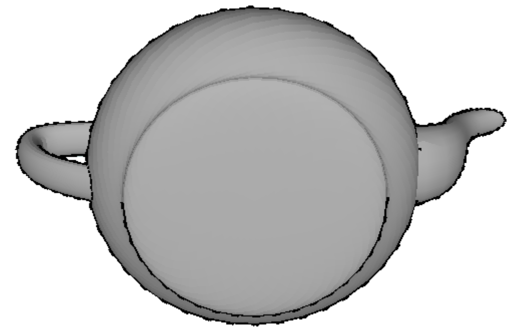
Statistics



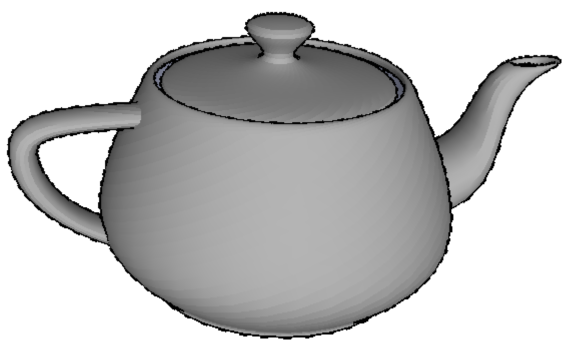
A



B

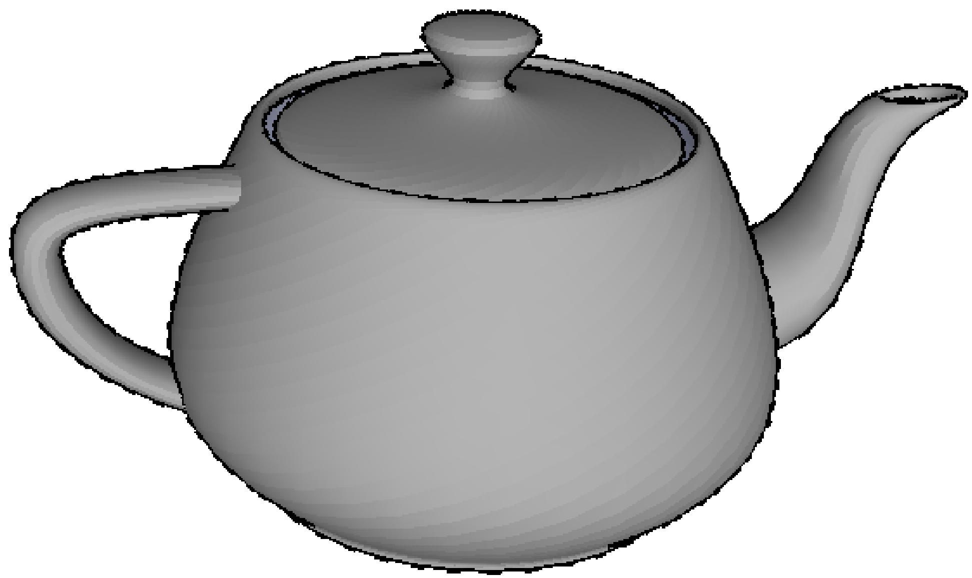


C



D

Why this position?



How to get to this position

Video

Principal component analysis

Finds the most meaningful basis to re-express a noisy, garbled data set.

Hopefully filters out the noise revealing any/some hidden dynamics.

- Face recognition
- Image compression

PCA method

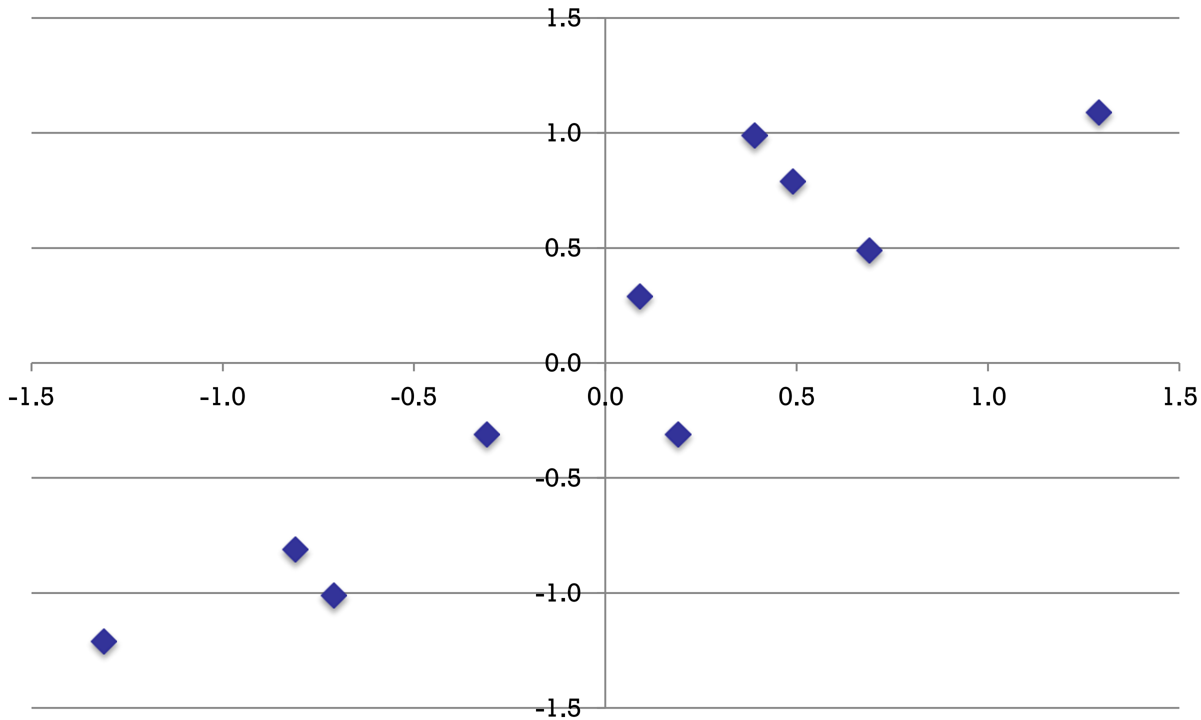
Step 1: Get hold of some data

Step 2: Subtract the mean for each variable

This makes the mean of each variable = 0

x	y
2.5	2.4
0.5	0.7
2.2	2.9
1.9	2.2
3.1	3.0
2.3	2.7
2.0	1.6
1.0	1.1
1.5	1.6
1.1	0.9

x'	y'
0.7	0.5
-1.3	-1.2
0.4	1.0
0.1	0.3
1.3	1.1
0.5	0.8
0.2	-0.3
-0.8	-0.8
-0.3	-0.3
-0.7	-1.0



Step 3: Find the covariance matrix

$$\mathbf{C} = \begin{pmatrix} cov(x,x) & cov(x,y) \\ cov(y,x) & cov(y,y) \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 0.616556 & 0.615444 \\ 0.615444 & 0.716556 \end{pmatrix}$$

$x'y'$
0.3381
1.5851
0.3861
0.0261
1.4061
0.3871
-0.0589
0.6561
0.0961
0.7171
0.6154444

x'^2	y'^2
0.4761	0.2401
1.7161	1.4641
0.1521	0.9801
0.0081	0.0841
1.6641	1.1881
0.2401	0.6241
0.0361	0.0961
0.6561	0.6561
0.0961	0.0961
0.5041	1.0201
0.616556	0.716556

Step 4: Calculate the eigenvectors and eigenvalues of the covariance matrix

Eigenvalues:

$\lambda_1 = 0.04948$

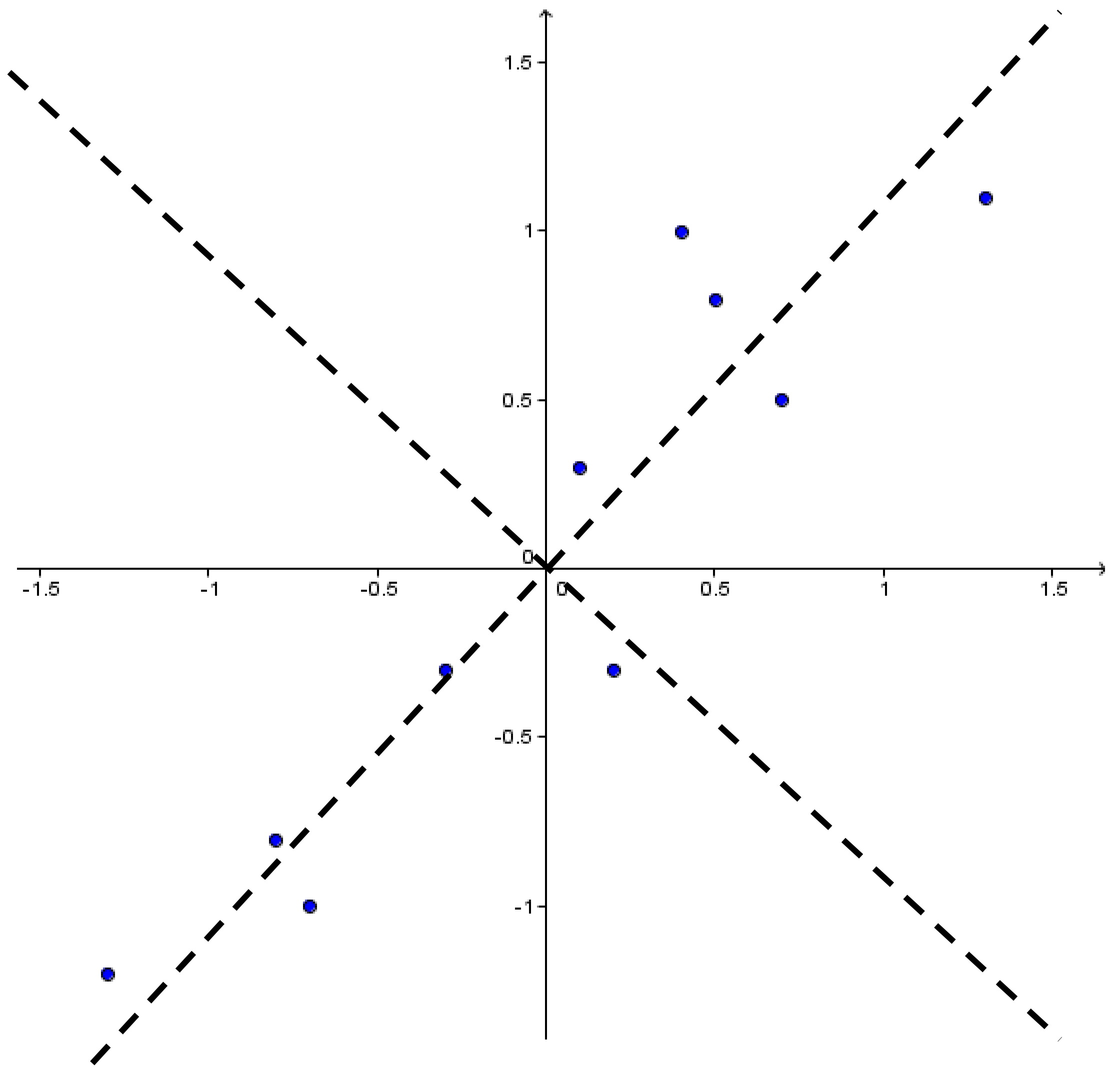
$\lambda_2 = 1.28402$

Eigenvectors (normalised):

$$\mathbf{v}_1 = \begin{pmatrix} -0.73518 \\ 0.67787 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} -0.67787 \\ -0.73518 \end{pmatrix}$$

What does this mean?



- The two variables increase together – covariance matrix shows this

- Eigenvectors provide information about the patterns in the data
  - One goes through the middle of the points - like a line of best fit. Shows the two data sets are related along that line
  - Other shows that all the points follow the main line, but are off to the side of it by some amount

Step 5: Transform the data so that it is expressed in terms of the two lines

The eigenvector with the highest eigenvalue is the *principal component* of the data set.

Ignore the components of lesser significance

This results in some information loss but not much if the eigenvalue was small

The final data set has one less dimension than the original data.

Use a *feature matrix* F where

$$\mathbf{F} = (eigenvectors)$$

In this case $\mathbf{F} = \begin{pmatrix} -0.67787 \\ -0.73518 \end{pmatrix}$

Transformed data is found using

$$(data\ matrix)\mathbf{F}$$

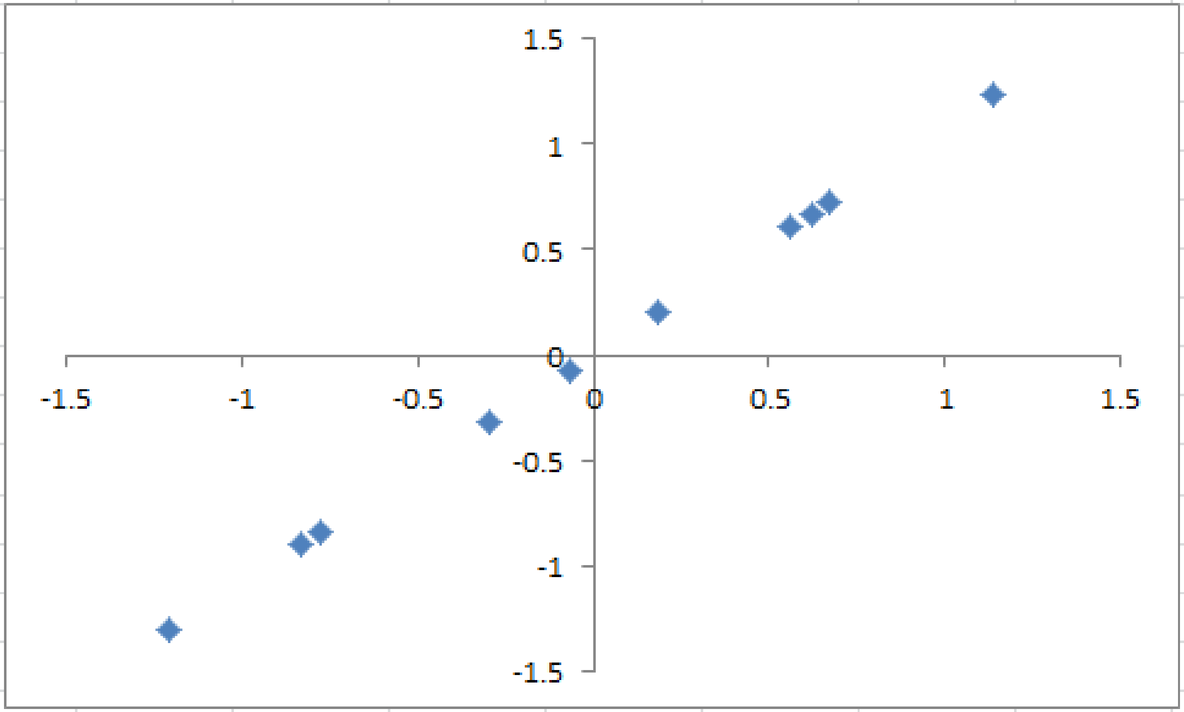
x''
-0.82797
1.777578
-0.9922
-0.27421
-1.6758
-0.91295
0.09911
1.144571
0.438046
1.22382

To show the information that has been lost

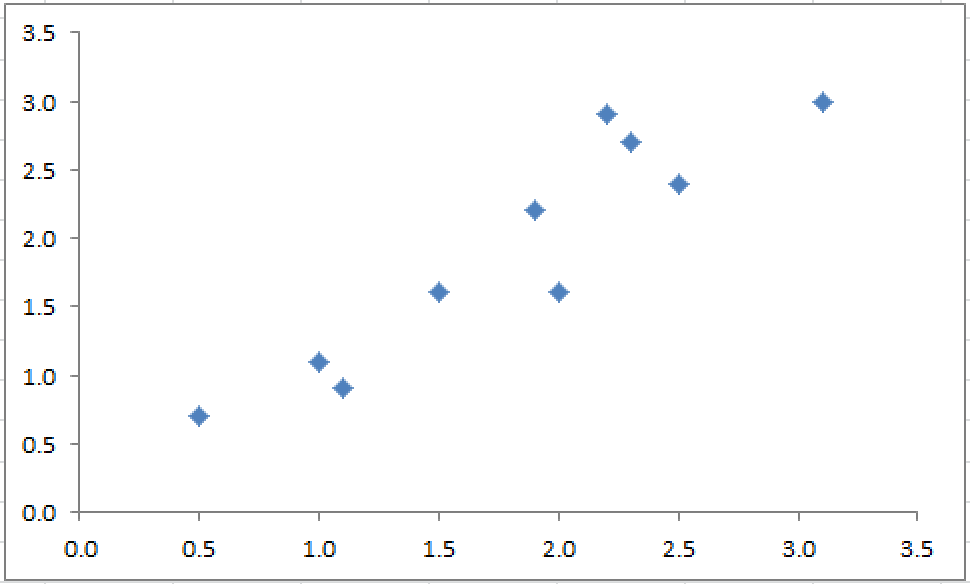
Use

$\mathbf{F}^{T}(adjusted\ data)$

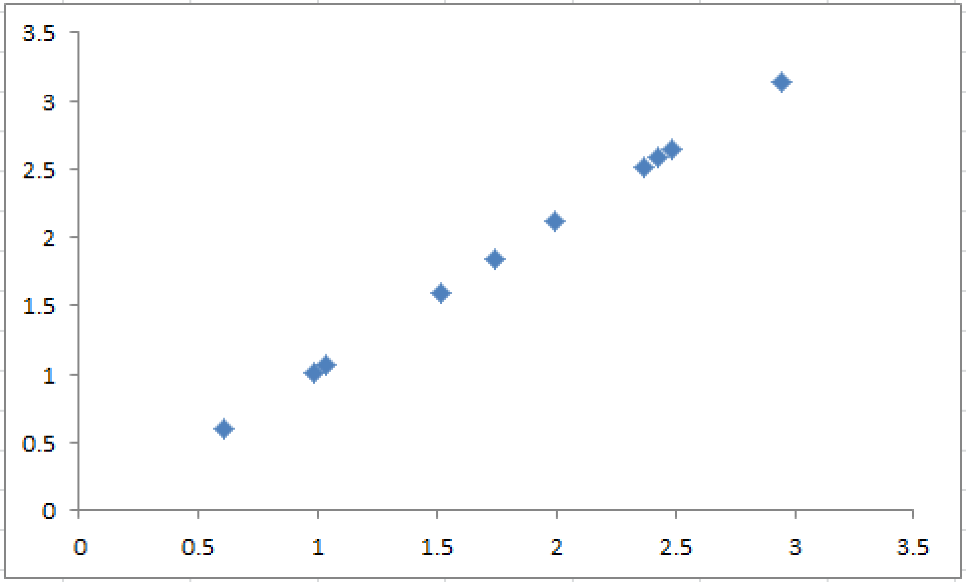
x'	y'
0.561255	0.608706
-1.20497	-1.30684
0.672581	0.729444
0.185879	0.201594
1.135974	1.232014
0.61886	0.671181
-0.06718	-0.07286
-0.77587	-0.84147
-0.29694	-0.32204
-0.82959	-0.89973



Original data



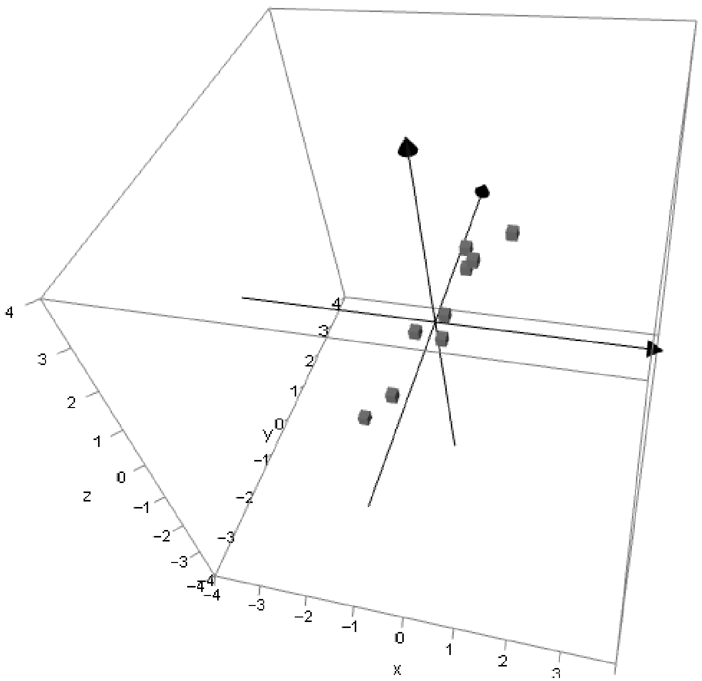
add the means

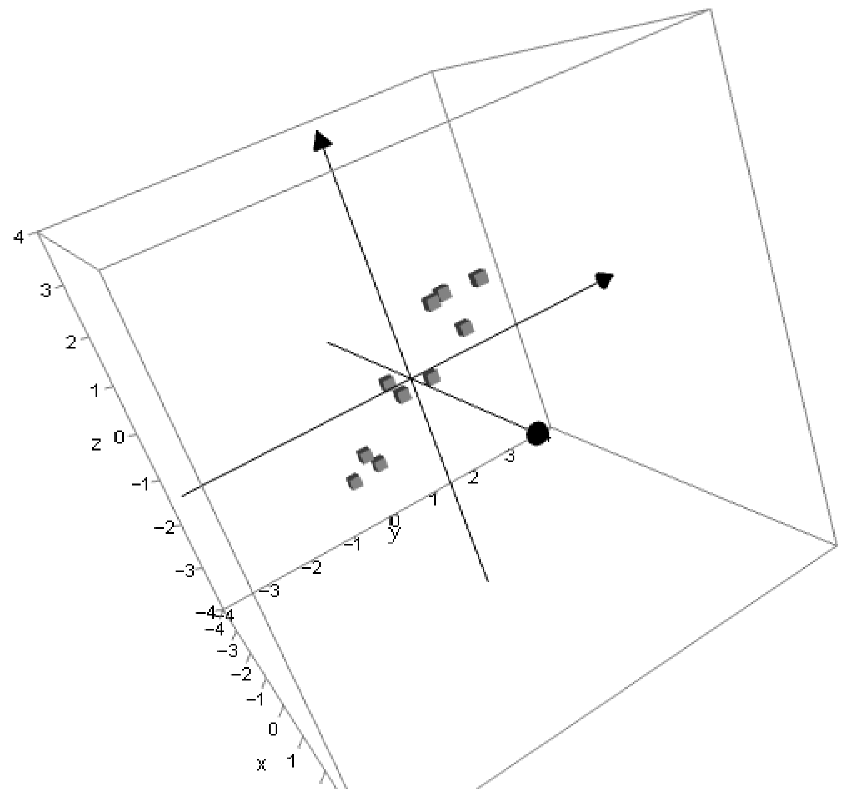


In 3D

x	y	z
2.5	2.4	3.1
0.5	0.7	0.2
2.2	2.9	2.4
1.9	2.2	1.8
3.1	3.0	3.3
2.3	2.7	3.1
2.0	1.6	1.8
1.0	1.1	0.5
1.5	1.6	1.9
1.1	0.9	0.8

x	y	z
0.7	0.5	1.2
-1.3	-1.2	-1.7
0.4	1.0	0.5
0.1	0.3	-0.1
1.3	1.1	1.4
0.5	0.8	1.2
0.2	-0.3	-0.1
-0.8	-0.8	-1.4
-0.3	-0.3	0.0
-0.7	-1.0	-1.1


x
y
z


x
y
z

Eigenvalues

$$\lambda_1 = 2.47218, \quad \lambda_2 = 0.06757, \quad \lambda_3 = 0.03437$$

$$\mathbf{v}_1 = \begin{pmatrix} 0.48845 \\ 0.51988 \\ 0.700809 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0.130045 \\ -0.837538 \\ 0.530677 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0.862847 \\ -0.168078 \\ -0.476701 \end{pmatrix}$$