Further Mathematics

- Linear qualifications, separate from AS/A level Maths
- 50% of A level Further Maths content fixed: 50% options
- AS Further Maths: up to 70% options
- AS Further Maths must be co-teachable with AS Maths (this is not easy)

It is hard to design a coherent AS Further Maths course which depends only on the content of AS Maths
Core Content: Further Maths

- **Proof**: by induction
- **Complex Numbers**: up to de Moivre, \( n \)th roots
- **Matrices**: transformations, 3 by 3 to solve equations
- **Algebra and functions**: summing series using differences, Maclaurin
- **Calculus**: improper integrals, mean value, volumes, more partial fractions, inverse trig
- **Vectors**: lines, planes & intersections, scalar product, distances
- **Polar coordinates**: area
- **Hyperbolic Functions**: calculus including use of inverse functions in integration
- **Differential equations**: second order, SHM, simultaneous 1st order, modelling with DEs

This is 50% of the A level
Core Content: AS Further Maths

- AS Further Maths must include some **complex numbers**, some **matrices**, relationship between roots and coefficients of a polynomial
- This counts for 20%; exam board chooses another 10% from the core A level content on the previous slide
- This leaves room for 70% options
Optional content

• What are the rules?
  – build on mechanics or statistics
  – new applications
  – extend the core content

• ALCAB and Decision maths
  “There is potentially a place in further mathematics for a serious strand of mathematics based on discrete mathematics and this could be considered as an additional strand alongside mechanics and statistics. However, this will require scrutiny to ensure that it will be perceived as a valuable part of further mathematics.”
# Assessment Objectives

<table>
<thead>
<tr>
<th>AO</th>
<th>Description</th>
<th>AS Maths</th>
<th>A level Maths</th>
<th>AS Further Maths</th>
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<td>AO1</td>
<td>Use and apply standard techniques</td>
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<td>AO2</td>
<td>Reason, interpret and communicate mathematically</td>
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<td>25%</td>
<td>At least 10%</td>
<td>At least 15%</td>
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<tr>
<td>AO3</td>
<td>Solve problems within mathematics and in other contexts</td>
<td>20%</td>
<td>25%</td>
<td>At least 10%</td>
<td>At least 15%</td>
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</table>
Opportunities and threats

- What do we gain and what do we lose by AS and A level Further Mathematics becoming linear qualifications, separate from Mathematics?
- How can we make the most of what we gain, and mitigate against what we might lose?
Desirable feature of a FM spec?

- What are the features of new AS/A level Further Mathematics qualifications which you would like to see?
- Think about teaching and about promoting it to potential students.
If you want to teach A level Further Maths across two years, parallel to A level Maths, then:

- Imagine the two years split into 6 teaching slots
- The mandatory Core pure paper takes 3 slots, one in Y12 and two in Y13
- There are three slots for optional units
Choose one major option and one minor option

Major options
- Mechanics major
- Statistics major

Choose three minor options

Minor options
- Mechanics minor
- Statistics minor
- Modelling with algorithms
- Numerical methods
  - Extra pure
  - Further pure with technology
MEI AS Further Mathematics

Choose two AS options

AS Further Maths works – anything you can sensibly teach in Year 12 of the two year A level, you can take as an AS unit.

AS Further Maths takes up three teaching slots
• The mandatory Core pure paper takes 1 slot; it’s the same content as one-third of the A level Core pure unit
• There are two slots for two optional units

AS options
• Mechanics a
• Statistics a
• Modelling with algorithms
• Numerical Methods
• Mechanics b
• Statistics b

AS
• Suitable for Year 12. Same content as A level minor option.
• Suitable for Year 13. Same content as second half of A level major option.
<table>
<thead>
<tr>
<th>A level Further Maths</th>
<th>Papers</th>
<th>Length</th>
<th>Raw marks</th>
<th>Proportion of A level (after scaling)</th>
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<td>AS options</td>
<td>Mech a; Stats a; MwA; NM; Mech b; Stats b</td>
<td>1 hr 15 mins</td>
<td>60</td>
<td>33(\frac{1}{3})%</td>
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Pure content

• National compulsory content 50% of A level Further Mathematics.
• At least 30% of AS Further Mathematics must be based on the compulsory content – we have made it one third.
Mechanics/Statistics

• For A level Further Maths you can do
  – no mechanics or
  – one-sixth mechanics (minor option) or
  – one-third mechanics (major option)

• For AS Further Maths you can do
  – no mechanics or
  – one-third mechanics or
  – two-thirds mechanics
Proposed model

• What features do you like and which do you dislike?
Mechanics

• Dimensional analysis
• Forces, moments and equilibrium
• Work, energy, power
• Momentum & impulse: direct impact collisions
• Centre of mass: system of particles

• Oblique impact collisions
• Circular motion
• Hooke’s law
• Vectors and variable forces: projectile up plane, SHM
• Centre of mass using calculus
Statistics

- Sampling
- Discrete random variables: uniform, Poisson, geometric
- Bivariate data: correlation and regression
- Chi-squared tests for contingency table and goodness of fit
- Bayes’ theorem
- Continuous random variables
- Confidence intervals based on Normal and $t$-distributions
- Wilcoxon test for single sample, compared with corresponding $t$-test and Normal test
- Simulation
Technology: confidence intervals

- Using $t$ or Normal distribution, as appropriate, to produce and interpret single sample, two-sample and paired sample confidence intervals

A Normal probability plot
The closer the points to a straight line the better a Normal distribution fits

A confidence interval based on the $t$ distribution, assuming the underlying distribution is Normal
Hui’s score is the total of 10 dice. What is the probability that her score is greater than 35?

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</table>
Modelling with algorithms

- Network & network algorithms
- Algorithms
- Linear programming: solving using graphical approach, simplex and software
- Reformulating network problems as LPs and using software to solve
Journey through Modelling with Algorithms

Model a problem with a network

What problems can be modelled as a network?
What is an algorithm? Does the algorithm give an optimal solution? How do you know? How does it scale up? Complexity.

Use an algorithm to find (e.g.) shortest path and interpret

Different problems can be modelled with the same network
Dijkstra, longest path, Prim, Kruskal, network flows
Algorithms for sorting and packing

For larger, authentic problems, reformulate as a linear programming problem
Investigate LP problems

Different problems can be modelled with the same network
Dijkstra, longest path, Prim, Kruskal, network flows
Algorithms for sorting and packing

Investigate LP problems
Algorithm

Does the algorithm give an optimal solution? How do you know?

How does it scale up? Complexity.

Use an algorithm to find (e.g.) shortest path and interpret

For larger, authentic problems, reformulate as a linear programming problem

Use simplex solving routine in software, and interpret output

network flows

Algorithms for sorting and packing

Investigate LP problems graphically

Integer variables

How does simplex work?
The new topics

- Reformulating a network problem as an LP

Minimise

\[2AB + 4BD + 4AC + 2CD + BC + CB\]

subject to

\[AB + AC = 1\]
\[AB + CB - BC - BD = 0\]
\[AC + BC - CB - CD = 0\]
\[BD + CD = 1\]

Find the shortest path from A to D
AB is a variable. It takes the value 1 if the path from A to D uses the arc from A to B. Otherwise it takes the value 0.
So for the path A to B to D, \(AB = 1\), \(BD = 1\), \(AC = CD = BC = CB = 0\)
The new topics

• Using software to solve an LP

Minimise

\[2AB + 4BD + 4AC + 2CD + BC + CB\]

subject to

\[AB + AC = 1\]
\[AB + CB - BC - BD = 0\]
\[AC + BC - CB - CD = 0\]
\[BD + CD = 1\]

\[AB = 1\]  \[BD = 0\]  \[AC = 0\]  \[CD = 1\]  \[BC = 1\]  \[CD = 0\]

Objective = 5
Numerical Methods

- Content as for current Numerical Methods
- No coursework
- Interpretation of output from spreadsheets will be expected
Topics in Numerical methods

• Use of technology
  – spreadsheets in the classroom; calculators

• Dealing with errors
  – how they arise, propagate and can be analysed

• Solution of equations
  – 5 methods, failure, order of convergence; relaxation

• Numerical differentiation
  – forward difference and central difference

• Numerical integration
  – midpoint, trapezium, Simpson’s

• Approximation to functions
  – Newton, Lagrange
What’s new?

• No coursework, so clearer expectations about the use of technology
• One small topic – relaxation
• Clearer descriptions in the spec of (e.g.)
  – order of convergence and order of method
  – what error analysis is expected
  – notation for Simpson, trapezium and midpoint rules
• Some modelling in exam questions
Spreadsheets in the exam

The spreadsheet printout shows the application of the secant method starting with \( x_0 = 0 \) and \( x_1 = 1 \). Successive approximations to the root are in column E.

(ii) What feature of column B shows that this application of the secant method has been successful? \([1]\)

(iii) Write down a suitable spreadsheet formula to obtain the value in cell E2. \([2]\)
A new topic - relaxation

For the iteration

$$x_{n+1} = g(x_n)$$

the relaxed iteration is

$$x_{n+1} = (1 - \lambda)x_n + \lambda g(x_n)$$

$$x_{n+1} = \exp(-x_n^2)$$

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Extra Pure

• Choose three topics from six
  – Combinatorics
  – Recurrence relations
  – Sets, logic and Boolean algebra
  – Groups
  – Multivariable calculus
  – Vectors and matrices
Further pure with technology

- **Investigation of curves**
  - use a graph plotter to explore curves, a computer algebra system (CAS), to explore tangents, length of curves

- **Number theory**
  - use simple programming skills to investigate results in number theory

- **Differential equations**
  - use a graph plotter to draw tangent fields, CAS to produce analytical solutions where they exist, a spreadsheet to use numerical methods

Recommend Geogebra & Python
In all its courses, MEI believes it is important to:

- promote mathematical thinking
- present mathematics in an interconnected way
- provide resources that promote mathematical learning
- support teachers in embedding new ideas in their classrooms
MEI
Innovators in Mathematics Education

Mathematics in Education and Industry

Over 50 years at the forefront of Mathematics Education

http://www.mei.org.uk/2017-mei-specification

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