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The Ancient Greek Mathematicians

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References and resources
The St Andrew’s MacTutor History of Mathematics Archive
Either search for this or enter via www-groups.dcs.st-and.ac.uk
This is a brilliant source of biographical material about mathematicians
with extensive references and links

A History of Greek Mathematics Vols 1 and 2; TL Heath
These books are quoted by everyone and major works about the Ancient
Greek Mathematicians. They are available free online in scanned versions. I
have found the Cornell University Library archive very good but
recommend the pdf not the Kindle versions. He wrote other comprehensive
works such as, The works of Archimedes.

Proofs Without Words: Classroom Resource Materials/ Number 1; The
Mathematical Association of America; isbn 978-0-88385-700-7
A book I strongly recommend for personal pleasure as well as class use.

Greek Numbers and Arithmetic – Math TAMU
An interesting and brief outline of how the Greeks represented numbers.
Searching the above title will take you to the article on the Texas A&M
University site.

Big Bang Simon Singh; Harper Perennial; isbn 978-0-00-715252-0
This very readable and informative book includes a very accessible account
of the development of scientific ideas that includes the calculation by the
ancient Greek mathematicians of the diameters of the Earth, Moon and Sun
and the distances between them.
The ancient Greek mathematicians

When one looks at the dates of these mathematicians (please see pages 21 and 22), two things that immediately catch one’s attention. The first is that they worked over a long period of time; from Pythagoras to Apollonius is about 400 years and from Pythagoras to Heron over 500 years. The second is that many of the best known worked a very long time ago at about the time of Alexander the Great and classical Athens. It is rather amazing that we know anything about them, let alone so much. They lived in and visited places all over the eastern Mediterranean in Italy, Sicily, Egypt, Asia Minor, mainland Greece and the Ionian and Aegean islands. Their contribution to the development of mathematics was enormous and they not only provided the starting point for the study of mathematics, they made giant strides along the journey.

One of the difficulties we have when we try to follow the work of the ancient Greeks is that they did not write and visualise mathematics in a way that is familiar to us. When one reads articles about them, it soon becomes apparent that it is hard to get to grips with exactly what they knew, in what form they knew it and how they reasoned. Understanding what they did is made more difficult because most articles about them are expressed in terms of modern ideas and modern notation which have benefited from another 2000 years of development and which would have had no meaning for the ancient Greeks. Taking Archimedes as an example. The consensus opinion is that a very high proportion of his work has proofs and methods that are entirely his own; he is widely regarded as being the greatest mathematician of ancient times and one of the greatest of all times. However, it is not always clear as to which of his theorems were completely original and, as it were, out of a blue sky, and in which ones he was making a contribution, albeit massive and in the form of a brilliantly convincing proof, to results already around (perhaps even from Babylon 1000 years earlier) or being developed at the time, mostly in Alexandria. It is also the case that he will never have written down any of his results in anything like the way we know them today.

We shall start by trying to enter a little into the ways that ancient Greeks wrote down and thought about mathematics

They had rather inconvenient ways of writing whole numbers and fractions.

They had no algebra as we would use it and so no formulae,

therefore they had no concept of coordinate geometry (not suggested by Descartes until 1637);

Definitions of figures such as a circle, parabola, ellipse and hyperbola would be in terms of the geometry of slices through a cone.

Results were given in terms of ratios of quantities and relationships were given a physical meaning by use of geometry; for instance

the area of a parallelogram given as being the same as that of a rectangle with the same base and height

the surface area of a sphere given as being 4 times the area of a great circle of the sphere
The ancient Greek mathematicians

the area of a circle given as the area of a right-angled triangle with one non-hypotenuse side the radius of the circle and the other the circumference of the circle

the volume of a cone given as one third of that of the prism on the same base and with the same perpendicular height

what we would write as \( ab = c^2 \), where \( c \) is called the mean proportional of \( ab \), might be considered as \( a \) and \( b \) being the lengths of the sides of a rectangle with the same area as a square of side \( c \)
or expressed geometrically as one of the intersecting chords properties of a circle

The method of **exhaustion** appears in several forms in ancient Greek mathematics. The essence of the method is to argue that if from any quantity at least half is removed and then this process is repeated each time removing at least half of the quantity remaining, then it is possible eventually to reduce the quantity remaining to less than any stated amount.

The Greeks debated the concept of a limit but their proofs only ever used processes with a finite number of steps. In order to complete a proof of a proposition, instead of saying they were **taking a limit** to find a quantity \( Q \), they instead showed that giving \( Q \) any value less than or greater than the value we would call the limit would lead to a contradiction.

In this session, we shall use modern terms and notation. In some cases we shall express ‘old’ proofs in modern terms, even though these modern ways of expressing ideas and developing them can be used to produce much more efficient methods of proof.

We shall now look at some of the prominent Greek mathematicians. In most cases there will be a brief biography followed by an account of some of their major ideas and some suggestions for some student activities. Your handouts have more detail than we shall have time for in the session and they include acknowledgements and suggestions about sources for further information.
The ancient Greek mathematicians

**Pythagoras** of Samos
~569 to ~475 BCE

Pythagoras is, perhaps, the earliest person we would describe as a *mathematician*. He is an extremely important figure but we know relatively little about his life and teachings. We have none of his writings and there is no specific piece of work that we can attribute to him; although there is some agreement about the main events of his life there is little that is certain.

We know that Pythagoras was influenced by other philosophers and that he went to Egypt in about 535 BCE. It is likely he extended his knowledge of geometry in Egypt and certainly adopted some customs of the priests such as secrecy, and refusal to eat beans or wear items made from animal skins.

After various travels during which he developed his ideas about mathematics and philosophy, Pythagoras founded a philosophical and religious school in Croton, on the heel of Italy. Because of the secrecy and commitment to collegiate working at the school, it is hard to know exactly what was done by Pythagoras and what was done by his followers but it is clear that they were not trying to solve ‘problems’ in the modern sense but were interested in the principles of mathematics, the concept of number, the concept of mathematical figures (e.g. the triangle) and the idea of proof. With our familiarity with abstractions, it is hard to appreciate just how important and difficult all of this was.

From his observations in music, mathematics and astronomy, Pythagoras developed the notion that all relations could be reduced to the relations between numbers. When he referred to the ratios between numbers, he meant ratios between whole numbers. He also thought that numbers had personalities (masculine or feminine, perfect or incomplete, beautiful or ugly).

The theorem we know that bears his name was known to the Babylonians 1000 years earlier but he may have been the first to prove it. Of course, to the Pythagoreans, the theorem would have been about the areas of squares constructed on the sides of a triangle, not about numbers that have been multiplied by themselves. According to T.L. Heath, other things known by the Pythagoreans were
- the sum of the interior angles and exterior angles of a polygon (as \(2n - 4\) and 4 right angles respectively)
- how to construct figures of given area
- how to use geometry to solve equations such as \(a(a - x) = x^2\)
- the 5 regular solids and the constructions of the first 3 of them
- the existence of irrational numbers

This last result is especially important. It is also interesting that it is most likely to have been discovered and proved by someone other than Pythagoras because of his belief that all relations were reducible to those between whole numbers.

It is worth mentioning some of the above when dealing with Pythagoras’ Theorem (especially if proving it geometrically), Pythagorean Triplets and the existence of irrational numbers.
The ancient Greek mathematicians

Students might be interested in some of the examples in *Proofs without Words* that show ways of establishing Pythagoras' Theorem.

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---adapted from the *Zhou bi suan jing*  
(author unknown, circa BCE 200?)

——Bhaskara (12th century)

---based on Euclid's proof

——Heron Perigal (1873)
The ancient Greek mathematicians

Eudoxus of Knidos
408 to 355 BCE

It seems that Eudoxus is in many ways one of the most original and important mathematicians of ancient times; he is regarded as the greatest mathematician of his time and second only to Archimedes amongst the early Greek mathematicians. Contemporary and later Greek mathematicians held his work in the highest regard and that is a view that continues until the present day.

He came from Knidos in Asia Minor, was widely travelled and studied in Sicily, Egypt and Athens. He founded his own highly regarded and successful school and many of his followers went with him on a second trip to Athens where he was at Plato’s Academy. It is suggested that Plato was not entirely pleased at the success of Eudoxus’ school; Eudoxus seems not to have had a high regard for Plato’s analytic abilities, his own mathematical skills being far superior.

Eudoxus’ work in many fields, including astronomy, includes some remarkably sophisticated ideas. His contributions to mathematics are remarkable. Notable among these are

- making the method of proof by exhaustion rigorous and going on properly to prove for the first time some results already in long use (most importantly and for which he is most celebrated and remembered) his work on proportion

Here is a little more about his work on proportion.

Before Eudoxus, definitions of what was meant by the comparison of the size of two ratios, including their equality, relied on the ratios being only of rational numbers. The discovery of irrational numbers by the Pythagorean school left these methods incomplete. This mattered a great deal as ancient Greek mathematicians calculated not with quantities and equations as we do but expressed the relationship between quantities as proportions.

Eudoxus developed a way to deal with the comparison of proportions which is as generally admired in modern mathematics as it was in the ancient world. All the ancient mathematicians seem simply to have adopted his ideas. The most celebrated book on mathematics probably of all time is Euclid’s Elements and this deals with proportion in Book V according to Eudoxus’ ideas.

Eudoxus first defines how ratios may be compared and then defines equality in a way that allows the ratios to include irrational numbers. In modern notation this is

\[ a : b \text{ and } c : d \text{ are equal (where } a, b, c, d \text{ are possibly irrational) if for every possible pair of integers } m \text{ and } n, \]
\[ \text{if } ma < nb \text{ then } mc < nd; \]
\[ \text{if } ma = nb \text{ then } mc = nd; \]
\[ \text{if } ma > nb \text{ then } mc > nd. \]
The ancient Greek mathematicians

Plato and Aristotle

With Socrates, who is the protagonist of many of Plato’s works, Plato and Aristotle are amongst the most influential philosophers of ancient times and, indeed, of all times.

The influence of Plato and Aristotle on the development of mathematics is very great but is not to be seen in the form of mathematical discoveries bearing their names. Their contributions were to the development of the philosophy of mathematics and to the important position that the study of mathematics had in their schools, Plato’s Academy and Aristotle’s Lyceum. For this reason we shall give them only a small part in the session.

Plato 427 to 347 BCE
Plato’s association with the 5 regular solids, called the Platonic solids, is not because of his discoveries about them but because he associated them with the 4 elements; earth with the cube, fire with the tetrahedron, air with the octahedron, water with the icosahedron and the whole universe with the dodecahedron.

A key part of Plato’s philosophy was his Theory of Forms. Objects in mathematics are perfect forms; for instance a line has length but no breadth and the notion of equality. His view that the study of mathematics was the ‘finest training for the mind’ was highly influential for 2000 years and his Academy had over its door the inscription Let no one unversed in geometry enter here.

Plato’s emphasis on the idea of ‘proof’, and on accurate definitions and clear hypotheses laid the foundations for Euclid’s later work. Indeed, it seems that most of the important mathematical work in the 4th century BCE was done by Plato’s friends or pupils. These include the very highly regarded Eudoxus (q.v.).

Aristotle 386 to 322 BCE
Perhaps the greatest philosopher ever, he is a towering figure in every one of the many disciplines he studied and in many cases initiated.

Aristotle was aware of the discoveries of other mathematicians (for instance, Eudoxus again) and also the importance of these discoveries. He had a thorough grasp of elementary mathematics and believed mathematics to be of great importance as one of the 3 ‘theoretical sciences’.

His major contributions to mathematics were to the development of logic, the recognition of the importance of axioms and the use of mathematics in his illustrations of scientific method.
The ancient Greek mathematicians

Euclid of Alexandria
~325 to ~265 BCE

Although not the greatest ancient Greek mathematician, Euclid is probably the best known because of his book *The Elements*. This book is a collection of mathematical knowledge. Few, if any, of the results were discovered or *first* proved by Euclid but the organisation and many of the definitions are his and so, probably, are many of the proofs given. This book, quite incredibly, was at the heart of mathematics teaching for 2000 years; more than 1000 editions have been published since it was first printed in 1482.

We know little about Euclid except that he taught at Alexandria and, because of his knowledge of the work of Eudoxus, it seems he must have studied at Plato's *Academy*. Indeed, although the dates given above are now generally accepted, it has been suggested by some that he lived much earlier and by others he lived much later. It is now generally accepted that *The Elements* and other works attributed to *Euclid* were, perhaps with some additions, written by a historical character. However, it has been argued that Euclid was the leader of a group of mathematicians who continued to write under his name even after his death; it has also been argued that the Euclid of *The Elements* was not a historical character but the name adopted by a group of mathematicians at Alexandria who took the name from a Euclid of Megara who lived 100 years earlier.

What is in *The Elements*?
It starts with definitions and 5 postulates. The postulates are effectively ‘reasonable assumptions’ such as it is possible to draw a straight line between any two points; there are also some implicit assumptions.

The 5th postulate that says that one and only one line can be drawn through a point parallel to a given line is the most famous one because it was found in the 19th century that dropping this postulate allowed non-Euclidean geometries, which have interesting and important properties.

The 13 books cover the following
- Books 1 – 2: basic properties of triangles, parallelograms, rectangles etc
- Book 3: the circle
- Book 4: problems involving circles (perhaps mostly work from Pythagoras’ followers)
- Book 5: Proportion (as proposed by Eudoxus)
- Book 6: The work in Book 5 applied to plane geometry
- Books 7 – 9: Number theory (including Euclid’s algorithm for HCF in Bk 7)
- Book 10: Irrational numbers
- Books 11-13 three-dimensional geometry
The ancient Greek mathematicians

The following examples of Euclid’s work, in modern notation, are likely to be seen at some time by all students. Of course, the wording and presentation would have to take account of their mathematical maturity and ability.

Both use the result that any factor of \( M \) and of \( N \) must be a factor of \( |M - N| \).

1. Proposition: There are an infinite number of prime numbers

Suppose there are only \( n \) prime numbers.
Label these numbers \( p_1, p_2, \ldots, p_n \)

Let \( N = p_1 \times p_2 \times \ldots \times p_n \) and \( q = N + 1 \)
\( q \) is either prime or it is not

\( q \) is not on the list \( p_1, p_2, \ldots, p_n \), (it must be larger than any of them) so

if \( q \) is prime, the proposition is established.

If \( q \) is not prime then it must have a prime factor, say \( p \).
If \( p \) were on our list it would divide \( N \) (the product of all the primes \( p_1, p_2, \ldots, p_n \)).
But \( p \) also divides \( q \) so \( p \) divides \( N \) and \( N + 1 \). Hence \( p \) divides \((N + 1) - N = 1\).
But no prime number can divide 1, so \( p \) cannot be on the list.
Hence there must be another prime not on the list \( p_1, p_2, \ldots, p_n \) and the proposition is established.

2. Euclid’s algorithm to find the HCF of two numbers.

This can be shown nicely in a flowchart.
[Expressions in the chart like \( A - B \rightarrow A \) mean that \( A \) is replaced by the difference \( A - B \). Of course, if on first presentation the chart is shown without any reference to HCF in the terminating box, the students can be asked to find out what the process is doing.]

Example: Suppose \( A = 18 \) and \( B = 12 \).
\( 18, 12 \rightarrow 6, 12 \rightarrow 6, 6 \).
The ancient Greek mathematicians

Archimedes of Syracuse
287 to 212 BCE

Archimedes was born in 287 BCE and little of his life known for certain. He possibly studied with Eratosthenes and the successors of Euclid at Alexandria and later certainly corresponded with the Euclidean school there.

We have 9 of his treatises (some partial or derivative) and know of others now lost.

Archimedes is now regarded as the greatest mathematician of antiquity and one of the greatest of all time, anticipating the methods of calculus by 1900 yrs.

He was famous in antiquity mainly for his inventions such as: orreries; the water-lifting screw; catapults; pulley systems; the claw (ship shaker); using sunlight to burn ships etc. Some of these inventions may have taken on exaggerated effectiveness.

Archimedes produced much original work including in new fields. This included many original rigorous proofs of his own discoveries as well as results already 'known'.

We know he believed that the study of ‘pure’ mathematics was the only worthy pursuit.

What sort of things did he do?

Worked out a more accurate value of \( \pi \), with error bounds \( \frac{223}{71} < \pi < \frac{22}{7} \)

Developed the theory of
- Compound pulleys
- Levers
- Centres of gravity
- Hydrostatics, including Archimedes’ Principle and the principles behind floating
- Spirals

He also worked on
- his system for writing large numbers based on the myriad (10 000)
- his method of Mechanical Theorems
- areas and volumes of spheres, cones and paraboloids
- areas of segments of parabolas

Archimedes (aged about 75) was killed by a Roman soldier in 212 BCE at the siege of Syracuse during the 2nd Punic War.

Several of his best known results can form the basis for some interesting work for students at many levels
The ancient Greek mathematicians

For years 10/11

**The volume and surface area of a sphere.**

Archimedes rigorously established these as $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$ but not, of course, in this form. For instance, he would have said something like, 'the surface area of a sphere is 4 times that of one of its great circles'.

The result of which he said he was most proud was

> the volume and surface area of a sphere are two-thirds of the volume and surface area respectively of a cylinder enclosing the sphere

and he had something like the diagram carved on his tomb.

Note that the surface area of the cylinder includes the plane ends.

Find the volume and surface area of the cylinder that encloses a sphere of radius $r$ and show that the formulae above are consistent with Archimedes' assertion.

Archimedes derived his results for the volume and surface area of a sphere by considering frustums of cones.

**The frustum of a cone**

Show that the curved surface area of the frustum of the cone shown in the diagram is $\pi(r + R)L$.

Show also that the volume of the frustum is $\frac{1}{3}\pi h(r^2 + rR + R^2)$. 
The ancient Greek mathematicians

For years 12/13

*Areas of segments of a parabola (The quadrature of the parabola)*

Suppose a chord cuts off a segment of a parabola. In the diagram below, the chord AB cuts off the segment APB. P is the point on the parabola found as follows: M is the midpoint of AB. The line through M parallel to the axis of the parabola cuts the parabola at P.

Archimedes proved that the area of the segment APB is \( \frac{4}{3} \) times the area of triangle APB.

There are several interesting things students can do with this result.

1. *Use Archimedes’ result for the area of a segment of a parabola* to show his result that the area of a parabolic arch is \( \frac{2}{3} \times \text{width} \times \text{height} \).

2. *Using calculus*, establish Archimedes’ result that the area of a parabolic arch is \( \frac{2}{3} \cdot w \cdot h \).
In example 3, use the diagram at the top of page 13 and assume that the equation of the parabola is $y = x^2$.

3  (i) Show that $P$ is the point between $A$ and $B$ on the curve that is at the greatest perpendicular distance from the line $AB$.

(ii) Show that triangle $APB$ has the largest area of any triangle that has base $AB$ and its third vertex on the parabola between $A$ and $B$.

(iii) Show that the tangent to the parabola at $P$ is parallel to the line $AB$.

(iv) Assuming that the points $A$ and $B$ have $x$-coordinates $a$ and $b$ respectively, show that the area of triangle $APB$ is $\frac{1}{8} (b - a)^3$. Comment on the form of this expression.
The ancient Greek mathematicians

Eratosthenes of Cyrene
276 – 194 BCE

Eratosthenes studied in Cyrene (in modern Libya) and in Athens. In about 240 BCE he became the 3rd librarian of the famous library in Alexandria in the temple of the Muses.

He was regarded by his contemporaries as a scholar of great distinction in all branches of knowledge but not of the highest ability in any of them. He was nicknamed Pentathlos (an all-round athlete) and also Beta which suggests he was not thought to be the best in any of them. Eratosthenes corresponded with Archimedes and worked in several aspects of mathematics including geometry and number theory.

The Sieve of Eratosthenes is known by almost all secondary school children. Starting with 2, as many integers as are being investigated are listed in order (see the grid on page 23). Beginning with the start of the list (the number 2), the following procedure is followed:

- call the member of the list \( n \);
- if \( n \) has been crossed out move to the next member of the list, \( n + 1 \), and repeat the procedure;
- if \( n \) has not been crossed out, first cross out all the multiples of \( n \) in the list and then move to the next member of the list, \( n + 1 \), and repeat the procedure.

Eventually, the only members of the list not crossed out are prime numbers. [It is worth noting that if the largest member of the list is \( N \), one has only to use the procedure on integers less than \( \sqrt{N} \).] Students who have not seen this procedure could be asked to work out what it does.

Perhaps the most remarkable thing done by Eratosthenes was to use some clever reasoning to find out how to determine the circumference of the Earth and then use this knowledge to find the diameter of the Moon and the distance of the Moon from the Earth. Using an argument from Aristarchus of Samos (~310 to ~230 BCE), it was then possible to estimate the distance from the Earth to the Sun. The reasoning used was as follows.

From observations of ships ‘hull down’, the apparent shape of the moon and the shape of the Earth’s shadow on the moon, Eratosthenes thought the Earth was a sphere. (This view was widely held at the time and it chimed with the Earth being a ‘perfect’ shape.) Aristarchus believed this too and also speculated on a heliocentric solar system. Both of them believed that the Sun was a very long way from the Earth (and the stars even further) so that light from the Sun reaching the Earth could be regarded as being parallel to a line joining their centres.

Eratosthenes knew that at Syene (near present day Aswan in Egypt) at noon on the summer solstice, the Sun shone directly down one of the wells. Eratosthenes realised that this meant that the Sun was directly ‘overhead’ at that time and so the well was on a line joining the centres of the Sun and the Earth. He also knew that such a thing never happened at Alexandria and argued that this must be because of the curvature of the Earth.
The ancient Greek mathematicians

He knew the distance between Syene and Alexandria and so arranged for the angle of the Sun to the vertical to be measured at Alexandria at the time it was directly overhead at Syene. This was done using the shadow of a vertical stick. Please see Fig. 1. The calculation is now easy. The circumference of the Earth is to the distance of Syene from Alexandria as the measured angle of the Sun to the vertical is to a complete revolution.

He thought the angle $\alpha$ was $1/50^{th}$ of a complete revolution; we think he took the distance between Syene and Alexandria to be a distance equivalent to about 800 km and so he seems to have found the circumference of the Earth to be about 40 000 km. It is impossible to know just how accurate the estimate was because Eratosthenes gave his distances in stadia and we are not sure exactly how long a stadium was. However, using the values usually given to convert stadia to kilometres, it is usually agreed that the value for the circumference of the Earth was very good and was between about 40 000 km (using Egyptian stadia) and 46 000 km (using Olympian stadia). The modern value is 40 100 km.

Others had already found the relative sizes of the Moon and the Earth by the following argument. Consider a total eclipse of the Moon. Given that the Sun is a long way from the Earth and the Moon, the Earth’s shadow must be approximately cylindrical. Fig. 2 shows the case where the Moon passes through the centre of the Earth’s shadow; the ratio of the time that it takes for the Moon to go completely into the shadow to the time that it takes from just entering the shadow one side to just leaving the other is the ratio of the diameter of the Moon ($d$) to that of the Earth ($D$). So if it takes time $t_1$ to go from position (i) to position (ii) and time $t_2$ to go from position (i) to position (iii), we must have $d: D = t_1 : t_2$. Eratosthenes obtained about $1:4$. 

![Fig. 1](image-url)
The distance of the Moon to the Earth may be found by the following method.

Hold up a disc of diameter $h$ cm diameter at a distance $l$ cm so that it just covers the Moon’s disc. Using similar triangles, it follows that the ratio of $h : l$ must be the ratio of the diameter of the Moon, $H$, to the distance of the Moon from the Earth, $L$. This readily gives a ratio of about 1:100. See Fig. 3.

Using the circumference of the Earth to be 40 000 km the Moon’s diameter was found to be about 3200 km and its distance from the Earth about 320 000 km.
The ancient Greek mathematicians

The distance of the sun from the Earth was found using a method described by Aristarchus. He argued that when one could see a half-moon, the triangle formed by the Sun (S), Moon (M) and Earth (E) must have a right angle at M. At such a time one had 'only' to measure the angle MES and use the already calculated distance EM. It is difficult to measure this angle. He found it to be about 87° (instead of 89.85°) and so found the Sun to 20 times the distance to the Moon instead of 400 times the distance.

Finally, the diameter of the Sun may be found using the well-known observation that the Moon almost exactly obscures the sun during a total eclipse of the Sun in a method similar to the one described above and shown in Fig. 3 to find the distance to the Moon. In this application, you know $h$, $l$ and $L$ and then find $H$. 

![Diagram](image)
The ancient Greek mathematicians

**Heron** of Alexandria

~10 to ~75 CE

Some sources use the Hero form instead of Heron. Some articles on him wrongly say that he lived at about 150 BCE or at about 250 CE.

It seems that he taught at the library at Alexandria in the temple of the Muses and lectured on mathematics, physics, pneumatics and mechanics. The last of these had a theoretical part and also a practical part about metals, architecture, carpentry, painting and other manual skills.

We have a lot of his works which cover topics such as: use of theodolites in surveying; how to find the distance from Alexandria to Rome using the difference between the local times at which an eclipse of the moon is observed; mechanical devices and engines (including one like a jet) worked by air, steam or water; an automaton puppet theatre; engines of war including catapults; much on the geometry of plane figures and solids including areas and volumes; mirrors; some statics including levers and dynamics.

It is not clear which of the ideas in his books are original to him.

A very interesting section from one of his books is on page 24.

Here are two ideas about which he writes that are likely to be his and are commonly used in schools.

1. A formula for calculating the area of a triangle given the lengths of the 3 sides.

   If a triangle has sides of length $a$, $b$ and $c$ then the area, $A$, is given by
   
   $$A = \sqrt{s(s-a)(s-b)(s-c)},$$
   
   where $s$ is the semi-perimeter $\frac{1}{2}(a + b + c)$.

   Example: Given a triangle with sides 3, 5, 7.
   
   $$s = \frac{1}{2}(3 + 5 + 7) = \frac{15}{2} = 7\frac{1}{2}$$
   
   $$A = \sqrt{\frac{15}{2}(7\frac{1}{2} - 3)(7\frac{1}{2} - 5)(7\frac{1}{2} - 7)} = \sqrt{\frac{15}{2} \times \frac{9}{2} \times \frac{5}{2} \times \frac{1}{2}} = \frac{15\sqrt{3}}{4}$$

2. Heron was the first recorded to solve problems of the following type.

   If $A$ and $B$ are points on a plane on the same side of a line $l$ and $C$ is a point on the line then the shortest distance $AC + CB$ is obtained when $AC$ and $CB$ are equally inclined to $l$.

   This can be applied to lots of interesting problems. Many of these are about the time taken to travel from $A$ to $B$; as long as the travel is at constant speed, the shortest length must give also the shortest time. In one famous application, this result became part of Fermat’s *principle of geometrical optics* also known as the *principle of least time* (about 1655 CE) which is consistent with modern quantum mechanical descriptions of the behaviour of light.
The ancient Greek mathematicians

The following is an example adapted from a more general class investigation of Fermat’s principle.

Where should you place the pump?
The diagram shows the plan view of two greenhouses A and B on the same side of a straight river and a pump P on the river. Each greenhouse obtains water through a pipe joined directly to the pump.
The task is to position the pump so that the total length of the pipes is as small as possible for this case and to generalise the answer for any position of A and B.

The above diagram can be displayed and the students can have paper copies with A, B and the river marked (but not P).
Ask them for ideas about what is involved and how they might obtain an approximate answer. If the class do not engage readily with the problem, suitable prompting questions could be:

- Make one suggestion which would result in less pipe being used
- The pump P can be placed anywhere on the river. Are there points on the river which obviously cannot be at the best place?
- Suggest a suitable position to try for your first measurement

In a class discussion, the following might be established

- the pipes should be straight
- P must obviously lie between U and V
- the mid-point of UV is a possibility but even rough measurement shows better positions for P are to the left of it

The following also might be suggested

- the problem might (would) be easier if A and B were equidistant from the river
- it would be much easier if A and B were on opposite sides of the river

You might consider starting with direct measurement using thread and pins. With the thread attached to a pin at A, it could pass round a moveable pin at P and then pass round the pin at B with a single turn. A bit fiddly but it might help students see that: the string should be taut (and so the pipes should be straight); there is a minimum between U and V; (perhaps) notice how the angles between the thread and the line of the river change.

The result that P must be in the position that makes angles APU and BPV equal may be easily proved using simple geometry. The key step is to reflect one of the points A or B in the river. Say B reflects to B’. Take P to be the point where AB’ crosses the river. It is easily shown that PB = PB’ so AP + PB = AP + PB’. The result follow. Please see the demonstration in GeoGebra:
https://www.geogebra.org/m/C7R8amEK
### The ancient Greek mathematicians

### The dates of some Greek mathematicians and events

<table>
<thead>
<tr>
<th>Mathematician</th>
<th>Born</th>
<th>Died</th>
<th>Era</th>
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<tbody>
<tr>
<td>Pythagoras</td>
<td>~569</td>
<td>~475</td>
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<tr>
<td>Eudoxus</td>
<td>408</td>
<td>355</td>
<td>BCE</td>
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<tr>
<td>Plato</td>
<td>427</td>
<td>347</td>
<td>BCE</td>
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<tr>
<td>Aristotle</td>
<td>384</td>
<td>322</td>
<td>BCE</td>
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<tr>
<td>Euclid</td>
<td>~325</td>
<td>~265</td>
<td>BCE</td>
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<tr>
<td>Aristarchus</td>
<td>~310</td>
<td>~230</td>
<td>BCE</td>
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<tr>
<td>Archimedes</td>
<td>287</td>
<td>212</td>
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<tr>
<td>Eratosthenes</td>
<td>276</td>
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<tr>
<td>Apollonius</td>
<td>~262</td>
<td>~190</td>
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<tr>
<td>Heron</td>
<td>~10</td>
<td>~75</td>
<td>CE</td>
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<tr>
<td>Diophantus</td>
<td>~200</td>
<td>~284</td>
<td>CE</td>
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<tr>
<td>Pappus</td>
<td>~290</td>
<td>~350</td>
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<thead>
<tr>
<th>Event</th>
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<tbody>
<tr>
<td>1st Peloponnesian war</td>
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<td>Peloponnesian war</td>
<td>431</td>
<td>404</td>
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<td>1st Punic war</td>
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<tr>
<td>2nd Punic war</td>
<td>218</td>
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<td>BCE</td>
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<tr>
<td>3rd Punic war</td>
<td>149</td>
<td>146</td>
<td>BCE</td>
</tr>
<tr>
<td>Greco-Persian wars</td>
<td>499</td>
<td>449</td>
<td>BCE</td>
</tr>
<tr>
<td>– 1st Persian invasion of Greece</td>
<td>492</td>
<td>490</td>
<td>BCE (ended at the battle of Marathon)</td>
</tr>
<tr>
<td>– 2nd Persian invasion of Greece</td>
<td>481</td>
<td>479</td>
<td>BCE (ended at the battle of Platea)</td>
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<tr>
<td>Alexander the Great</td>
<td>356</td>
<td>323</td>
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<tr>
<td>– conquest of Persian empire etc</td>
<td>334</td>
<td>324</td>
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The ancient Greek mathematicians

BCE 3000 2000 1000 0 1000 2000 CE
- Great pyramid
- Old Babylonian Empire
- Ancient Greek Mathematicians

BCE 600 500 400 300 200 100 0 100 CE
- Pythagoras
- Plato
- Euclid
- Aristarchus
- Archimedes
- Eratosthenes
- Apollonius
- Punic wars
- Alexander the Great
- Greco-Persian wars
- Peloponnesian wars
- Much later
- Heron
- Diophantus
- Pappus
The ancient Greek mathematicians

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Heron's description of the ancient Babylonian method for finding square roots.

Since 720 has not its side rational, we can obtain its side within a very small difference as follows. Since the next succeeding square number is 729, which has 27 for its side, divide 720 by 27. This gives $26\frac{2}{3}$. Add 27 to this, making $53\frac{2}{3}$, and take half this or $26\frac{5}{6}$. The side of 720 will therefore be very nearly $26\frac{5}{6}$. In fact, if we multiply $26\frac{5}{6}$ by itself, the product is $720\frac{1}{36}$, so the difference in the square is $\frac{1}{36}$. If we desire to make the difference smaller still than $\frac{1}{36}$, we shall take $720\frac{1}{36}$ instead of 729 (or rather we should take $26\frac{5}{6}$ instead of 27), and by proceeding in the same way we shall find the resulting difference much less than $\frac{1}{36}$.