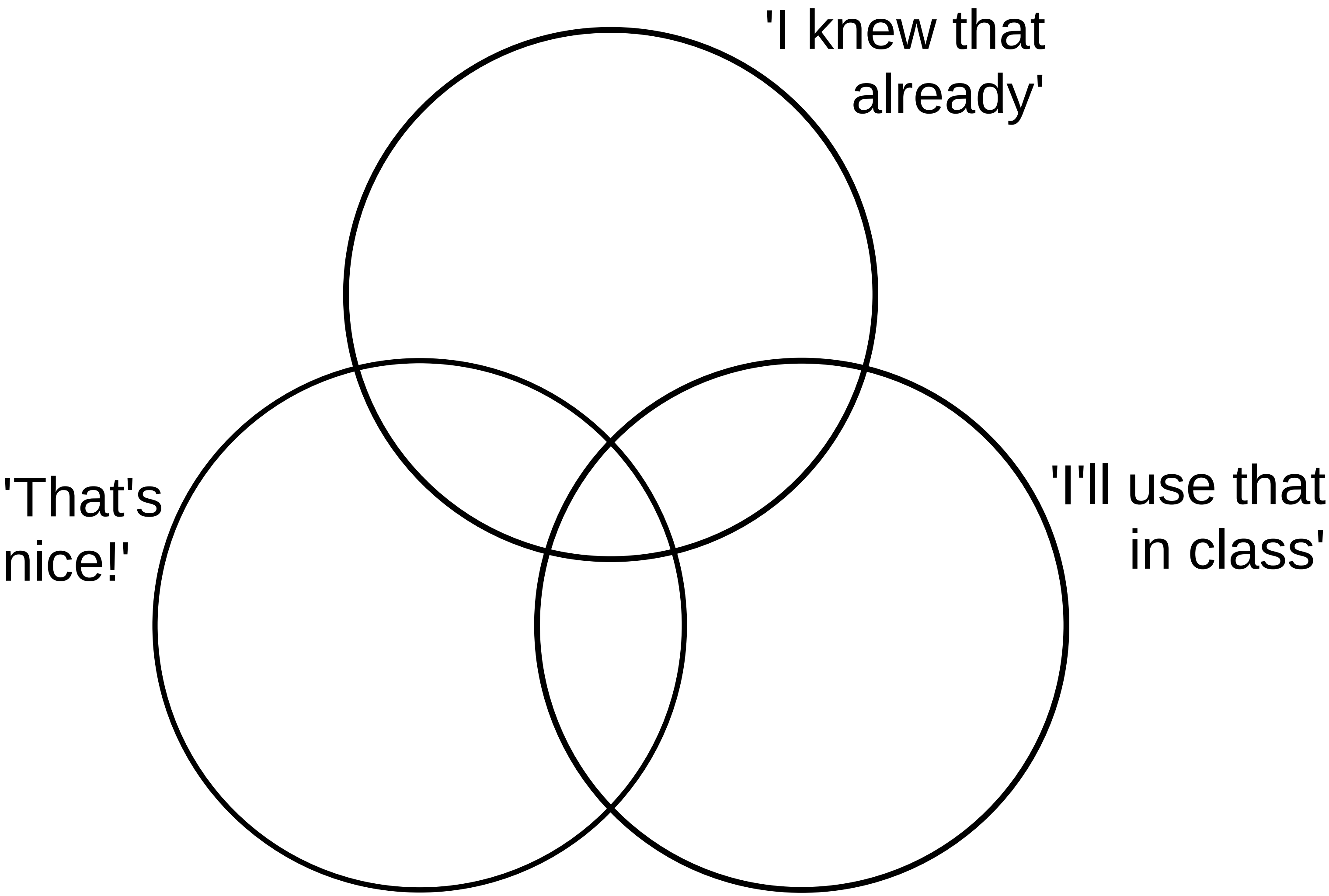
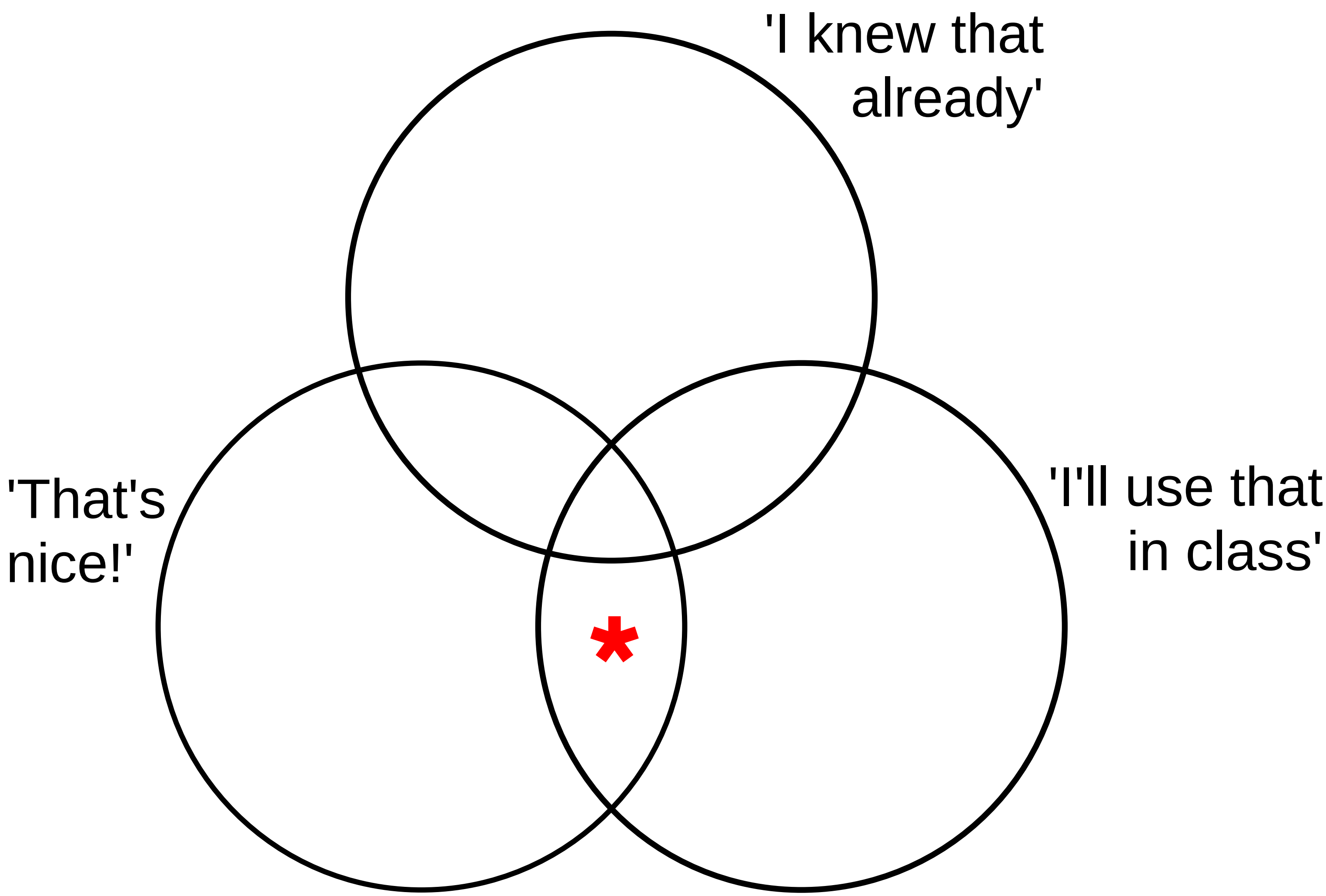
Deepening understanding in A level Mathematics through number

Prime factorisation, factorials, recurring decimals and the base 10 number system are all ideas that are met at KS3 and 4.

In this session we will look at how these ideas can be used in A level Mathematics, deepening both students' understanding of A level topics and their appreciation of number.


'I knew that already'
'That's nice!'
'I'll use that in class'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Proof Differentiation Integration

Geometric series Binomial theorem

Indices Logarithms Polynomials

I’m thinking of a quadratic function, $\mathrm{f}(x)$.

You give me an integer a and I’ll tell you $\mathrm{f}(a)$.

How many goes before you can tell me what my quadratic is?

Think of a polynomial function, $f(x)$,
in which all the coefficients
are integers between 0 and 9.

I give you an integer a and you tell me $f(a)$.

How many goes before I can tell you
what your polynomial is?

Prime then square

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, …

Make your own log tables

x	2	3	4	5	6	7	8	9	10	11
$\log(x)$									1	

Make your own log tables

$$2^{10} \approx 1000 \quad \Rightarrow \quad \log_{10} 2^{10} \approx 3 \quad \Rightarrow \quad \log_{10} 2 \approx 0.3$$

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x	2	3	4	5	6	7	8	9	10	11
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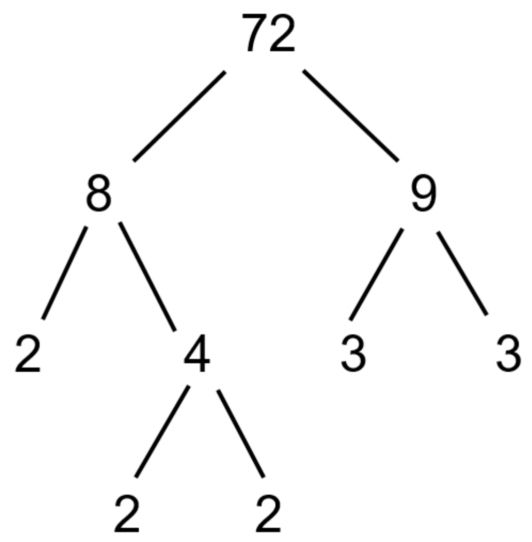
$$3^2 \approx 10 \qquad\qquad 3^4 \approx 2^3 \times 10$$



How many factors has 72?

And what do they add up to?

	1	2	2^2	2^3
1	1	2	4	8
3	3	6	12	24
3^2	9	18	36	72

	1	2	2^2	2^3
1	1	2	4	8
3	3	6	12	24
3^2	9	18	36	72

$$\left(1+2+2^2+2^3\right)\left(1+3+3^2\right)=15\times13=195$$

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Perfect numbers

$$6 = 1 + 2 + 3$$

$$28 = 1 + 2 + 4 + 7 + 14$$

$$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$$

Perfect numbers

$$6 = 2^{2-1}\left(2^2 - 1\right)$$

$$28 = 2^{3-1}\left(2^3 - 1\right)$$

$$496 = 2^{5-1}\left(2^5 - 1\right)$$

Euclid: If $2^p - 1$ is prime then $2^{p-1}\left(2^p - 1\right)$ is perfect.

Perfect numbers

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Euclid: If $2^p - 1$ is prime then $2^{p-1}\left(2^p - 1\right)$ is perfect.

Euler: Every even perfect number is of this form

‘Rule of 72’

A sum of money is invested at a rate of r% per year.

How many years before the sum doubles in value?

$$2 = (1 + r)^n$$

r	actual	72/r
1%	69.661	72
2%	35.003	36
3%	23.450	24
4%	17.673	18
5%	14.207	14.4
6%	11.896	12
7%	10.245	10.3
8%	9.006	9
9%	8.043	8
10%	7.273	7.2

'Rule of 72'

A sum of money is invested at a rate of r% per year.

How many years before the sum doubles in value?

$$2=(1+r)^n$$

$$2\approx 1+nr+\frac{n(n-1)}{2}r^2$$

$$2\approx 1+nr+\frac{(nr)^2}{2}$$

$$(nr)^2+2nr-2\approx 0$$

$$\Rightarrow nr\approx\sqrt{3}-1\approx 0.73$$

r	actual	72/r
1%	69.661	72
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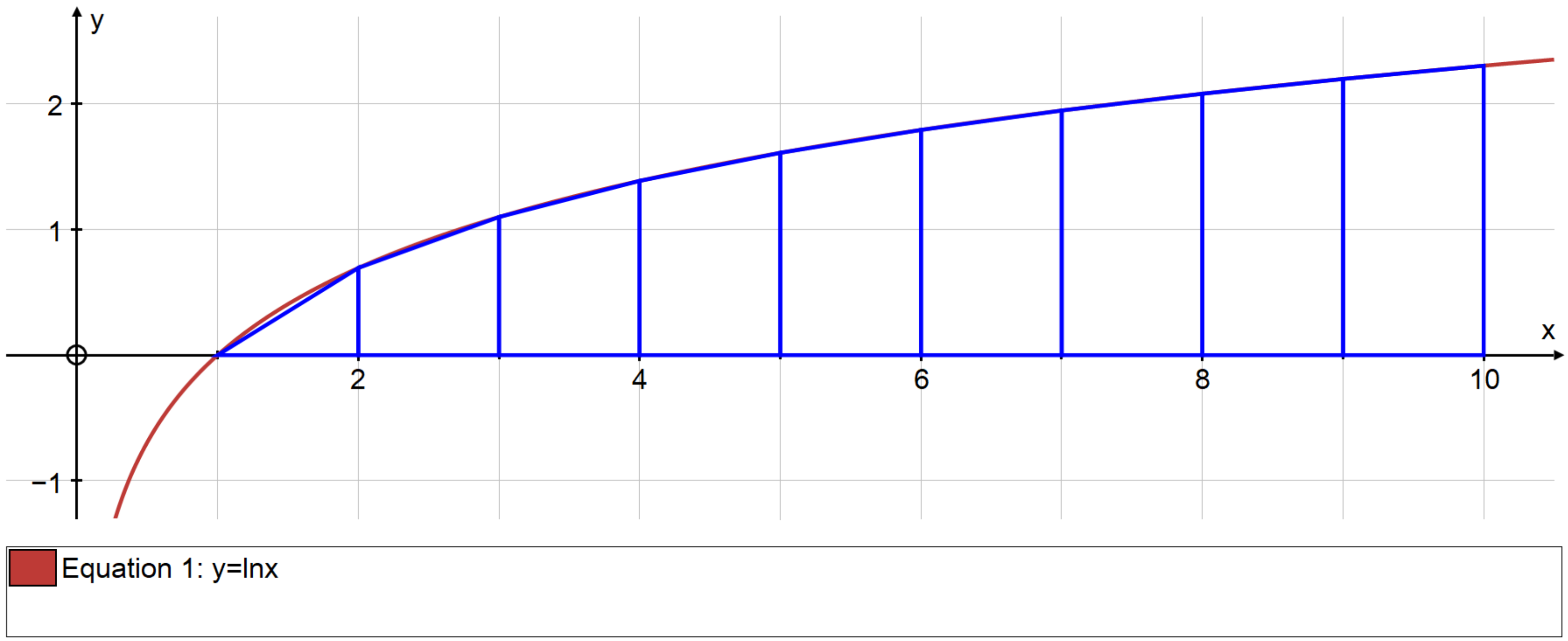
Roughly how big is 50! ?

(a) 10^{45} (b) 10^{55} (c) 10^{65} (d) 10^{75}

Approximating factorials



$$\int_1^n \ln x \, dx \approx \frac{1}{2}\left(\ln 1 + 2\left(\ln 2 + \ln 3 + \ldots + \ln(n-1)\right) + \ln n\right)$$

Approximating factorials

$$\frac{1}{2}\left(\ln 1+2\left(\ln 2+\ln 3+\ldots+\ln\left(n-1\right)\right)+\ln n\right)\approx\int_1^n \ln x\,\mathrm{d}x$$

$$\ln 2+\ln 3+\ldots+\ln\left(n-1\right)+\frac{1}{2}\ln n<\int_1^n \ln x\,\mathrm{d}x$$

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$$\ln 2 + \ln 3 + \ldots + \ln\left(n-1\right) + \ln n < \left(n\ln n - n + 1\right) + \frac{1}{2}\ln n$$

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$$\ln n! < \left(n + \frac{1}{2}\right)\ln n - n + 1$$

Approximating factorials

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$$\ln 2+\ln 3+\ldots+\ln\left(n-1\right)+\ln n<\left(n\ln n-n+1\right)+\frac{1}{2}\ln n$$

$$\ln n!<\left(n+\frac{1}{2}\right)\ln n-n+1$$

$$n!<\frac{n^{\left(n+\frac{1}{2}\right)}}{\mathrm{e}^{n-1}}=\mathrm{e}\sqrt{n}\left(\frac{n}{\mathrm{e}}\right)^n$$

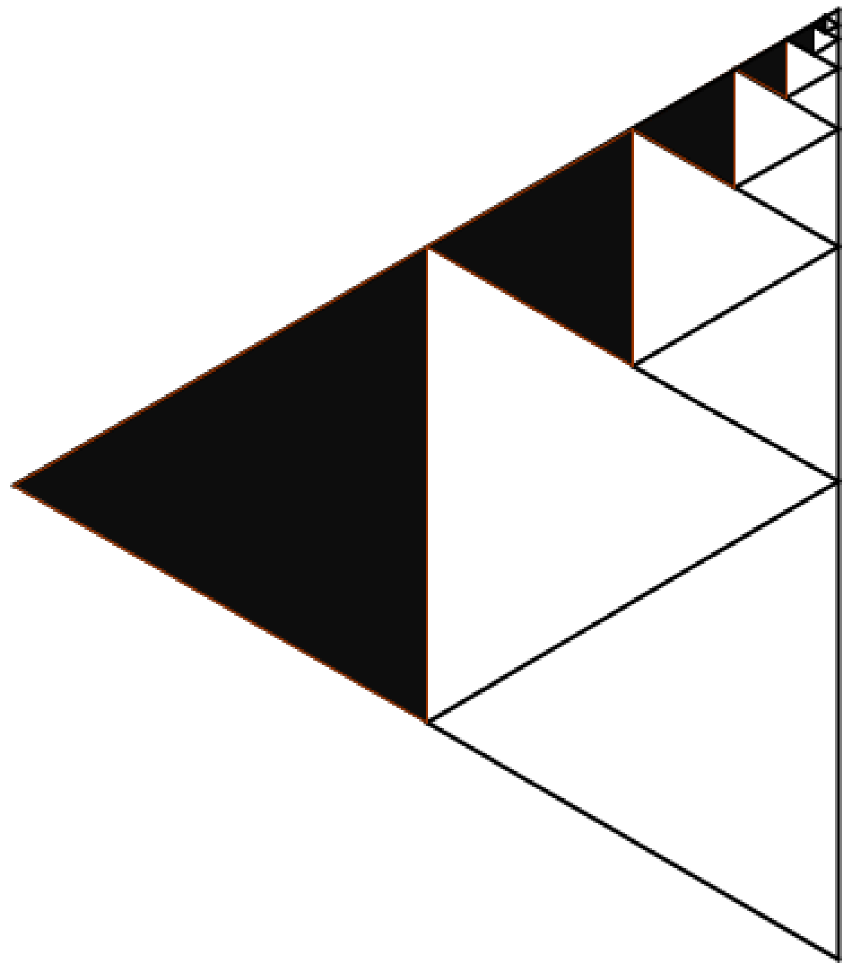
Stirling's Formula: $n! \approx \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n$.

n	$n!$	Our approximation	Stirling's approximation
5	120	128	118
10	3628800	3902561	3598696
15	1.31×10^{12}	1.41×10^{12}	1.30×10^{12}
20	2.43×10^{18}	2.63×10^{18}	2.42×10^{18}
50	3.04×10^{64}	3.29×10^{64}	3.04×10^{64}
100	9.33×10^{157}	1.01×10^{158}	9.32×10^{157}
150	5.71×10^{262}	6.19×10^{262}	5.71×10^{262}

Geometric series

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}$$

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$$



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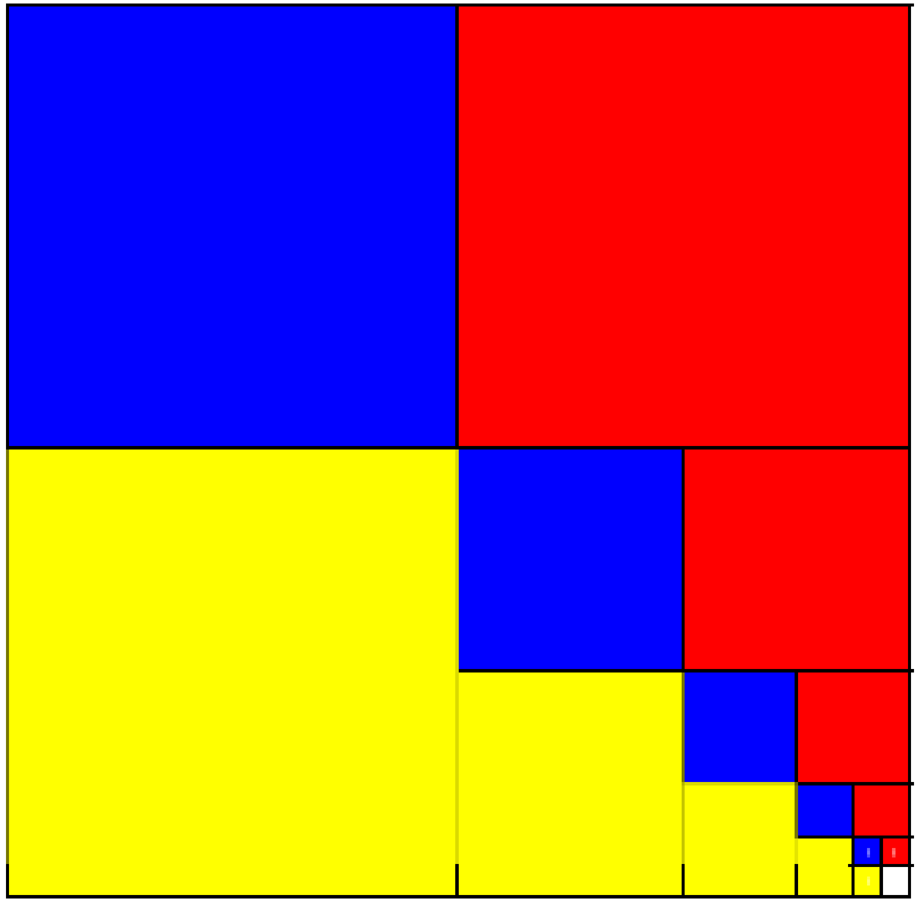
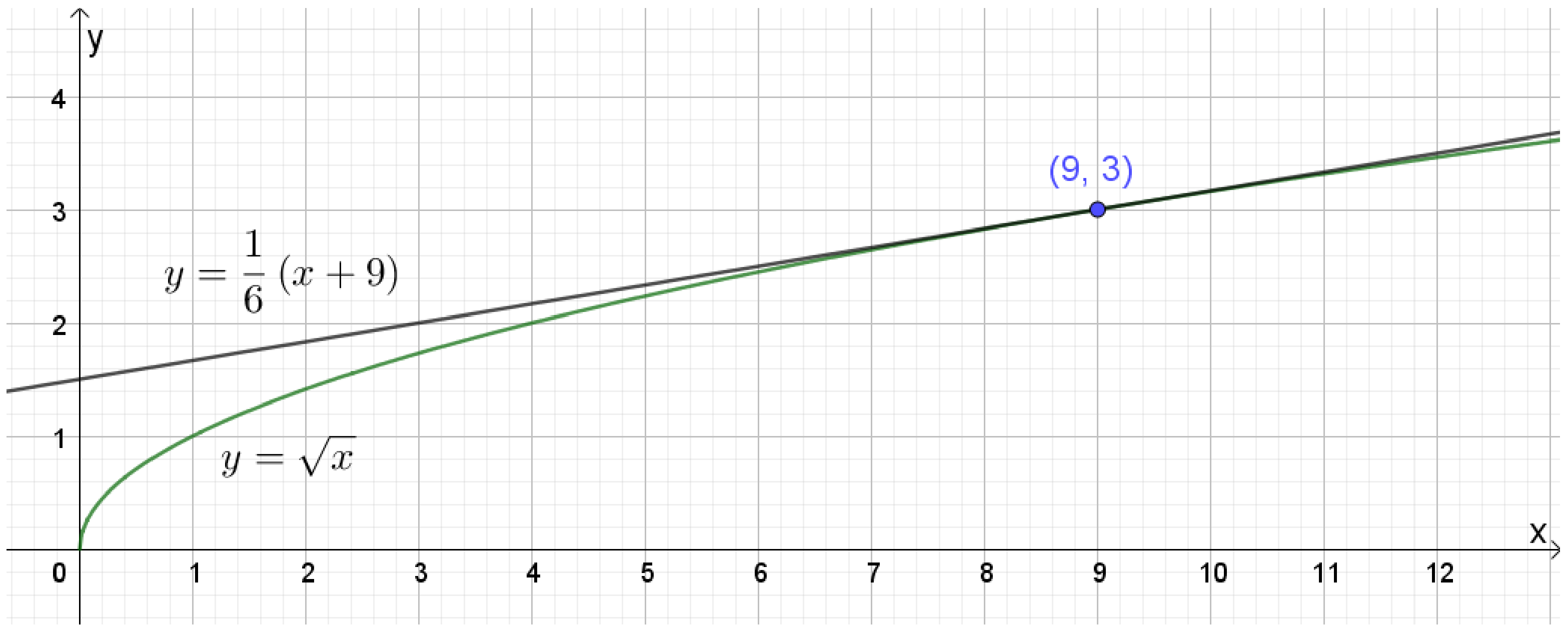
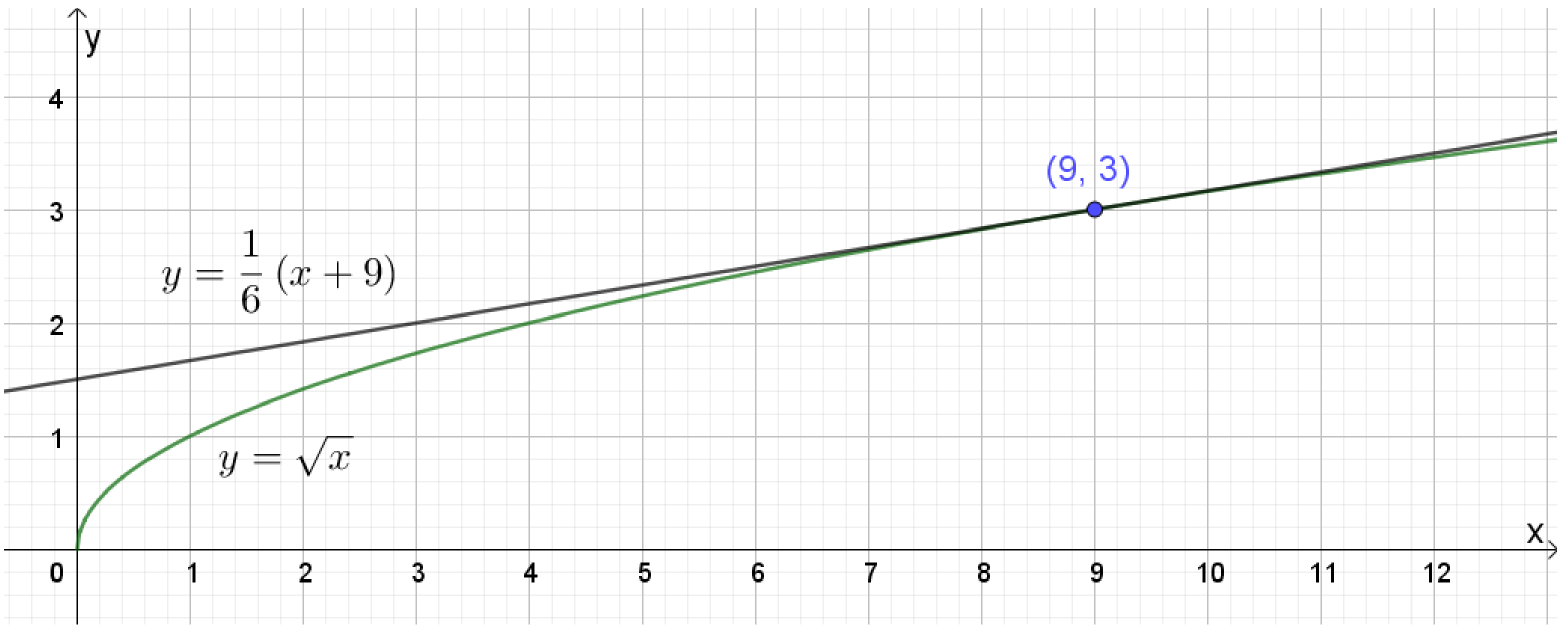


Approximating $\sqrt{11}$



Approximating $\sqrt{11}$



$$(3+x)^2 = 11$$

$$9 + 6x \approx 11$$

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The binomial theorem

$$2=\frac{2}{1}=\frac{8}{4}=\frac{18}{9}=\frac{32}{16}=\frac{50}{25}=\frac{72}{36}=\frac{98}{49}=\frac{128}{64}=\frac{162}{81}=\frac{200}{100}=\frac{242}{121}=\frac{288}{144}=\frac{338}{169}=\ldots$$

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$$\sqrt{2} = \sqrt{\frac{100}{49} \times \frac{98}{100}} = \frac{10}{7}\left(1 - \frac{1}{50}\right)^{\frac{1}{2}} = \frac{10}{7}\left(1 - \frac{1}{2 \times 50} - \frac{1}{8 \times 50^2} + \ldots\right)$$

The binomial theorem

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$$3=\frac{3}{1}=\frac{24}{8}=\frac{81}{27}=\frac{192}{64}=\frac{375}{125}=\frac{648}{216}=\frac{1029}{343}=\frac{1536}{512}=\frac{2187}{729}=\frac{3000}{1000}=\ldots$$

$$\sqrt[3]{3}=$$

The binomial theorem

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$$\sqrt[3]{3}=\sqrt{\frac{1000}{343}\times\frac{1029}{1000}}=\frac{10}{7}\left(1+\frac{29}{1000}\right)^{\frac{1}{3}}=\frac{10}{7}\left(1+\frac{29}{3\times 1000}-\ldots\right)$$

$$
\begin{array}{r l}
 & 1.\ 2\ 3\ 4\ 5\ 6\ 7\ 9\ 0\ 1\ 2\ldots \\
\hline
81 \,\big)\, & 100.{}^{19}0{}^{28}0{}^{37}0{}^{46}0{}^{55}0{}^{64}0{}^{73}0{}^{1}0{}^{10}0{}^{19}0\ldots.
\end{array}
$$

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}, \qquad -1 < x < 1$$

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$$1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{1}{(1-x)^2}$$

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$$1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{1}{(1-x)^2}$$

$$x = 0.1:$$

$$1 + 0.2 + 0.03 + 0.004 + \ldots = \frac{1}{0.9^2} = \frac{100}{81}$$

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$$\begin{array}{r} 0.\;0\;\;1\;\;1\;\;2\;\;3\;\;5\ldots \\ 89\,\overline{)\,1.{}^{1}0{}^{10}0{}^{11}0{}^{21}0{}^{32}0{}^{53}0\ldots} \end{array}$$

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\end{array}
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6 1 0

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20
21	22	23	24
25	26	27	28
29	30	31	32
33	34	35	36
37	38	39	40

Any questions please email Bernard.murphy@mei.org.uk

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