MEI Conference 2018

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Odd and interesting sequences

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@suedepom


ONE CHEESY TORTILLA CHIP...
ONE CHEESY TORTILLA CHIP...
TWO CHEESY TORTILLA CHIPS...
THREE CHEESY TORTILLA CHIPS...
FIVE CHEESY TORTILLA CHIPS...
EIGHT CHEESY TORTILLA CHIPS...
MATH GEEKS SHOULDN'T BE ALLOWED ANYWHERE NEAR CERTAIN FOODS.
WHAT'S WRONG WITH FIBONACHOS?
THERE'RE ONLY 12 LEFT...NOW WHAT?
www.foxtrot.com

Bored of Fibonacci? Try Lucas

- The French mathematician Edouard Lucas (1842-1891), investigated 2, 1, 3, 4, 7, 11, 18 ... using the rule of adding the latest two to get the next
- Can you spot the connection between Lucas numbers and Fibonacci numbers?

F_n	0	1	1	2	3	5	8	13	21	34
L_n	2	1	3	4	7	11	18	29	47	76

$$L_n = F_{n-1} + F_{n+1}$$

Lucas Numbers

n	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34
L_n	2	1	3	4	7	11	18	29	47	76

- Try adding alternate Lucas numbers. How does it relate to the Fibonacci numbers?

$$5F_n = L_{n-1} + L_{n+1}$$

For more about Lucas numbers try

http://www.maths.surrey.ac.uk/hosted sites/R.Knott/Fibonacci/lucasNbs.html

1, 11, 21, 1211, 111221,. . .

- What’s the next term?
- This is John Conway’s ‘Speak and Say’ sequence
- The sequence only contains the digits 1, 2, and 3, unless the starting number contains a larger digit or a run of more than three of the same digit
- The sequence grows indefinitely but here is one exception. Can you find the exception?

1, 11, 21, 1211, 111221,.

- Start with any digit *d* from 0 to 9. What do you notice?

  d, 1*d*, 111*d*, 311*d*, 13211*d*, 111312211*d*, 31131122211*d*, …

- If you start with any digit *d* from 0 to 9 then *d* will always be the last digit of the sequence.

A few odd and interesting facts

- The terms in the sequence eventually grow in length by about 30% per generation
- *If* L_n denotes the number of digits of the n^{th} term of the sequence, then

$$\lim_{n\to\infty}\left(\frac{L_{n+1}}{L_n}\right)=\lambda$$

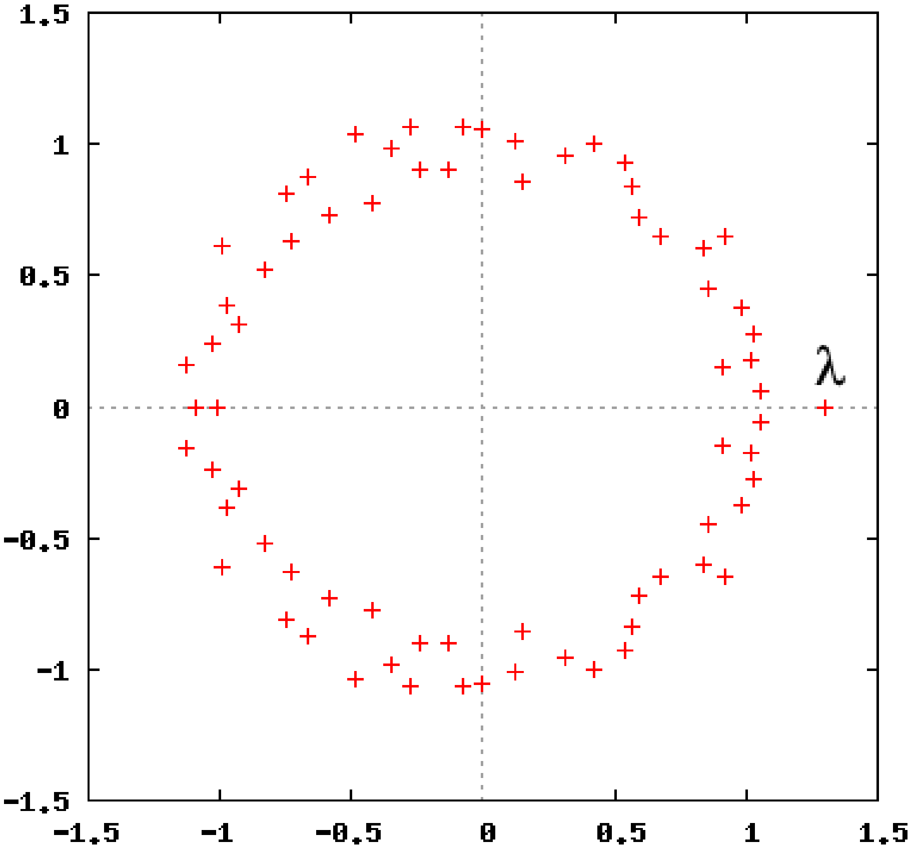
$$\text{where } \lambda = 1.303577....$$

*this number is known as **Conway's constant***

A few odd and interesting facts

- Conway and Richard Parker also showed that, no matter how you begin the sequence, this same limiting property holds for any sequence constructed by this rule
- Conway also showed that λ is a positive, real root of a polynomial of degree 71.

$$
\begin{aligned}
0 = {} & x^{71} - x^{69} - 2x^{68} - x^{67} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} - \\
& x^{60} - x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} - 3x^{53} - 2x^{52} + 6x^{51} + \\
& 6x^{50} + x^{49} + 9x^{48} - 3x^{47} - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42} + 8x^{41} - \\
& 5x^{40} - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} + x^{35} - 3x^{34} + 10x^{33} + x^{32} - 6x^{31} - \\
& 2x^{30} - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} - 3x^{25} + 14x^{24} - 8x^{23} - 7x^{21} + \\
& 9x^{20} + 3x^{19} - 4x^{18} - 10x^{17} - 7x^{16} + 12x^{15} + 7x^{14} + 2x^{13} - 12x^{12} - \\
& 4x^{11} - 2x^{10} + 5x^{9} + x^{7} - 7x^{6} + 7x^{5} - 4x^{4} + 12x^{3} - 6x^{2} + 3x - 6,
\end{aligned}
$$



Historical note

The limiting property (more generally, Conway's "Cosmological Theorem"*) was first proved in the late 1970's but the details of that original proof, were lost for some time.

In 1997, Shalosh, Ekhad and Zeilberger used a computer to check the theorem and recreated the proof.

**look it up*

https://en.wikipedia.org/wiki/Look-and-say_sequence
https://arxiv.org/pdf/math/9808077.pdf

Digit product sequences

- Let n be a positive integer, written in base 10
- f(n) = n + (the product of the non-zero digits of n).
- Write the first 12 terms with n=1

1, 2, 4, 8, 16, 22, 26, 38, 62, 74, 102, 104

108, 116, 122, 126, 138, 162, 174, 202, 206, 218, 234, 258, 338, 410, 414, 430, 442, 474, 586, 826, 922, 958, 1318, 1342, 1366, ...

- Who cares? It's just playing with the digits of values, big deal.

1, 2, 4, 8, 16, 22, 26 38, 62, 74, 102, 104

- Try again starting with 3. Do you notice anything?

3, 6, 12, 14, 18, 26, 38, 62, 74, 102, ...

- Starting from a different seed, the two sequences joined at 26.
- Paul Loomis did a lot of research on this. In 1989 he conjectured that starting from any positive integer, its sequence will join the sequence started from 1 eventually

- For any natural number *n*, the sequence merges with the sequences starting at 1

Numbers that take a long time to converge are called stubborn numbers. The first is 63 which merges on its 323rd term at 150,056

- Numbers that don’t occur in any previous sequence at called unattainables

There are an infinite number of unattainables

More at
https://facstaff.bloomu.edu/ploomis/sequences.html

MEI Mathematics Education Innovation

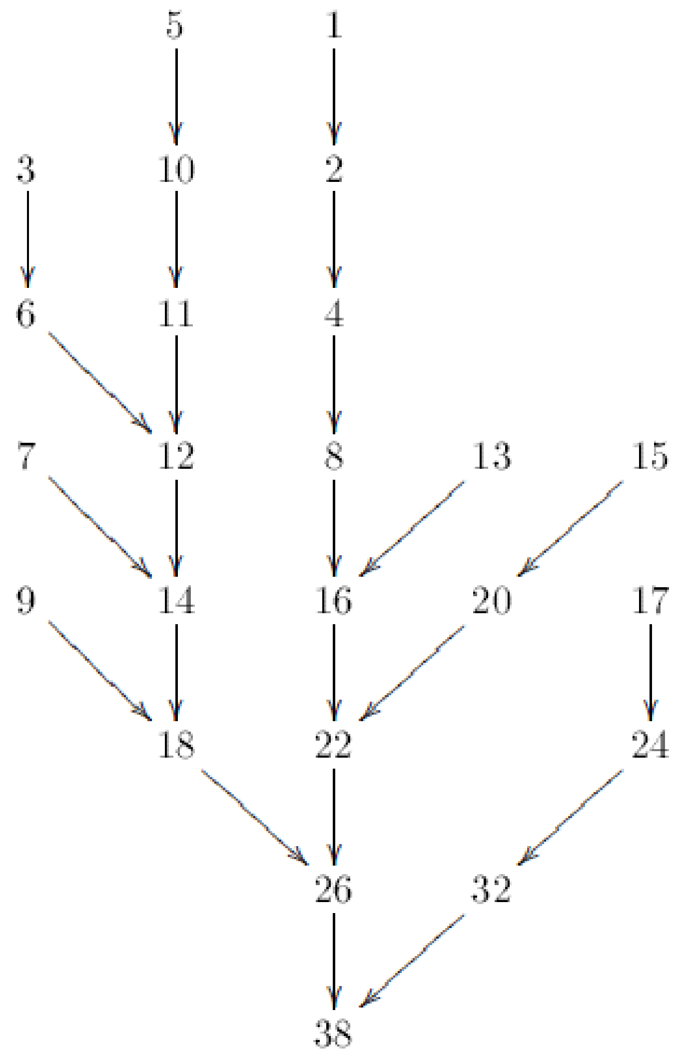


Fig. 1. Sequence tree in base 10

Ref: An Introduction to Digit Product Sequences, Paul Loomis, Department of Mathematics, Bloomsburg University, PA

More ideas

- Underground Maths
  https://undergroundmathematics.org/sequences
  - Change one thing
  - Sort it out
  - Square spirals and visual proof
- The Lazy Caterer's sequence
  https://en.wikipedia.org/wiki/Lazy_caterer%27s_sequence
- Nrich
  https://nrich.maths.org/public/search.php?search=sequences&filters[ks5]=1

Magic

- Write out the positive integers
- cross out the even numbers
- sum the remaining terms

1 ~~2~~ 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9 ~~10~~ 11 ~~12~~

1 4 9 16 25 36

or fluke?

Magic

- Write out the positive integers
- cross out the multiples of 3
- sum the remaining terms

1 2 ~~3~~ 4 5 ~~6~~ 7 8 ~~9~~ 10 11 ~~12~~ 13 14 ~~15~~

1 3 7 12 19 27 37 48 61 75

Magic

- cross out every second number from these sums
- sum the remaining terms

1 2 ~~3~~ 4 5 ~~6~~ 7 8 ~~9~~ 10 11 ~~12~~ 13 14 ~~15~~

1 ~~3~~ 7 ~~12~~ 19 ~~27~~ 37 ~~48~~ 61 ~~75~~

1 8 27 64 125

or fluke?

Moessner's magic

- No prizes for guessing what happens if you start with every fourth number

1 2 3 ~~4~~ 5 6 7 ~~8~~ 9 10 11 ~~12~~ 13 14 15 ~~16~~

1 3 ~~6~~ 11 17 ~~24~~ 33 43 ~~54~~ 67 81 ~~96~~

1 ~~4~~ 15 ~~32~~ 65 ~~108~~ 175 ~~256~~

1 16 81 256

Moessner’s theorem

- Alfred Moessner: known for his contributions to recreational maths; he wrote for *Scripta Mathematica*, a quarterly journal published by Yeshiva University in New York
- He published this conjecture in 1951 but never proved it
- It was proved by Oskar Perron – less than a year after its initial publication.
- There are several proofs and extensions of Moessner's theorem

Moessner's magic

- **What happens if you start by crossing out the triangle numbers?**

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Moessner's magic

- Delete the triangle numbers
- Delete the last number in each group and repeat

~~1~~	2	~~3~~	4	5	~~6~~	7	8	9	~~10~~	11	12	13	14	~~15~~
	(2)		6	~~11~~		18	26	~~35~~		46	58	71	~~85~~	
			(6)			24	~~50~~			96	154	~~225~~		
						(24)				120	~~274~~			
										(120)				

Sums of powers

$$\sum_{1}^{n} 1 = n$$

$$\sum_{1}^{n} r = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\sum_{1}^{n} r^2 = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{2}{12}n$$

$$\sum_{1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

Bernoulli numbers

$$\sum_{1}^{n} 1 = n$$

$$\sum_{1}^{n} r = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\sum_{1}^{n} r^2 = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{2}{12}n$$

$$\sum_{1}^{n} r^3 = \frac{1}{3}n^4 + \frac{1}{2}n^3 + \frac{3}{12}n^2 + 0n$$

$$\sum_{1}^{n} r^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{4}{12}n^3 + 0n^2 - \frac{1}{30}n$$

$$\sum_{1}^{n} r^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 + 0n^3 - \frac{1}{12}n^2 + 0n$$

Sums of powers

- The general pattern seems to be that there are formulas that look like:

$$\sum_{1}^{n} r^k = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \frac{k}{12} n^{k-1} + 0n^{k-2} + \ldots$$

- but the question remains, what patterns can we find in the coefficients on the right-hand side?
- The answer was first studied by the 17th century mathematician Johann Faulhaber, and later given a more thorough treatment in the 18th century mathematician Jacob Bernoulli.

Bernoulli numbers

$$\sum_{1}^{n} 1 = n$$

$$\sum_{1}^{n} r = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\sum_{1}^{n} r^2 = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{2}{12}n$$

$$\sum_{1}^{n} r^3 = \frac{1}{3}n^4 + \frac{1}{2}n^3 + \frac{3}{12}n^2 + 0n$$

$$\sum_{1}^{n} r^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{4}{12}n^3 + 0n^2 - \frac{1}{30}n$$

$$\sum_{1}^{n} r^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 + 0n^3 - \frac{1}{12}n^2 + 0n$$

$$B_0 = 1$$

$$B_1 = \frac{1}{2}$$

$$B_2 = \frac{1}{6}$$

$$B_3 = 0$$

$$B_4 = -\frac{1}{30}$$

$$B_5 = 0$$

Generalise

$$\sum_{r=1}^{n} r^k = \frac{1}{k+1}n^{k+1} + \frac{1}{2}n^k + \frac{k}{12}n^{k-1} + 0n^{k-2} + \ldots$$

$$\sum_{r=1}^{n} r^k = \sum_{r=0}^{k} \frac{B_r}{r!}\frac{k!}{(k-r+1)!}n^{k-r+1}$$

Where B_k are the Bernoulli numbers

Often called Faulhaber's formula

Bernoulli numbers

- B_k is the coefficient of $\frac{x^k}{k!}$ in the Taylor series expansion of $\frac{x}{e^x - 1}$

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_1 \frac{x^2}{2} + B_2 \frac{x^4}{4!} + \dots + B_n \frac{x^n}{(n)!}$$

Bernoulli numbers

- By substituting $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ substituting this into the denominator of $\frac{x}{e^x - 1}$

we get a strategy for calculating the Bernoulli numbers recursively: $\sum_{i=1}^{n} \binom{n+1}{i} B_i = 0, \quad \text{n} \geq 1$

This was used by Ada Lovelace to demonstrate the potential of Babbage's Analytical Engine

Bernoulli numbers

- The generating function is the exponential function
- They can also be defined by a contour integral
- Ramanujan wrote about them in his first paper for the Indian Mathematical Society
- The Bernoulli numbers also appear in the Taylor series expansions of tan and tanh, in the Euler–Maclaurin formula, and in expressions for certain values of the Riemann zeta function.

For a better explanation

Have a look at

https://www.youtube.com/watch?v=yGpkB2OoQjk

Conway paper

Some Very Interesting Sequences

John H. Conway and Tim Hsu

July 6, 2006

Download at
http://www.math.sjsu.edu/~hsu/pdf/sequences.pdf

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