MEI Conference 2018

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Vector geometry in FM

	AQA	Edexcel	MEI	OCR	WJEC
AS	Lines Distances (point/line, skew lines)	Lines Planes Distances	Planes	Lines Vector product	Lines Planes Distances
A level	Planes Distance point/plane Vector product		Lines Vector product Distances	Planes Distances	
Other		Vector product in FP1		More vec product in Add Pure	
Formula book	None	Point/plane	Skew lines Point/plane	Skew lines Point/plane	Skew lines Point/plane

Disclaimer!

This session will focus on understanding different methods and how they relate to each other.

- Check the details of the spec you teach and any teaching guidance provided by the awarding body
- Check the specimen papers and markschemes (and real papers once available)
- Consult the awarding body if in any doubt about appropriate methods

What is the vector product?

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{pmatrix}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$$

where θ is the angle between **a** and **b** and $\hat{\mathbf{n}}$ is a unit vector perpendicular to **a** and **b**

Key point: the vector product of **a** and **b** gives a vector which is perpendicular to both **a** and **b**

Vector geometry in FM

Problem 1

Finding the equation of a plane through three given points

Vector geometry in FM

Find the Cartesian equation of the plane through the points P (1, 2, 3), Q (1, 0, -1) and R (2, -1, 1)

What different approaches are there?

Vector geometry in FM

Find the Cartesian equation of the plane through the points P (1, 2, 3), Q (1, 0, -1) and R (2, -1, 1)

Use simultaneous equations

$$a + 2b + 3c = d$$

$$a - c = d$$

$$2a - b + c = d$$

Vector geometry in FM

Find the Cartesian equation of the plane through the points P (1, 2, 3), Q (1, 0, -1) and R (2, -1, 1)

Use the form r = a + λb + μc (a is a point on the plane, b and c are vectors in the plane)

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

(the λ vector is labelled $\overrightarrow{PR}$; the μ vector is labelled $\overrightarrow{QR}$)

Hence write down three equations and eliminate λ and μ

Vector geometry in FM

Find the Cartesian equation of the plane through the points P (1, 2, 3), Q (1, 0, -1) and R (2, -1, 1)

Find a vector perpendicular to two vectors in the plane using the scalar product

$$\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0 \qquad \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$$

Write plane in form $n_1x + n_2y + n_3z = d$ and use one of the points to find d

Vector geometry in FM

Find the Cartesian equation of the plane through the points P (1, 2, 3), Q (1, 0, -1) and R (2, -1, 1)

Find a vector perpendicular to two vectors in the plane using the vector product

$$\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

Write plane in form 4x + 2y - z = d and use one of the points to find d

Vector geometry in FM

Wolfram demonstrations

http://demonstrations.wolfram.com/ConstructingVectorGeometrySolutions/

Need to download the Wolfram CDF player

Then download the demonstration as CDF

Vector geometry in FM

Find the equation of a plane through three points A, B and C

What steps are involved? (Wolfram)

1. Construct two vectors in the plane (e.g. AB and AC)
2. Vector product of these two vectors
3. Plane through A (say) with normal from (2)

Vector geometry in FM

Problem 2

Finding the distance of a point from a plane

Vector geometry in FM

Find the distance from the point P (1, 2, 3) to the plane $2x - 3y + z = 4$

What steps are involved?

1. Line through P with direction vector **n**
2. Find M, the intersection of line and plane
3. Distance between P and M

Vector geometry in FM

Find the distance from the point P (1, 2, 3) to the plane 2x – 3y + z = 4

$$\mathbf{r} = \mathbf{p} + \lambda\hat{\mathbf{n}}$$

$$(\mathbf{p} + \lambda\hat{\mathbf{n}}).\hat{\mathbf{n}} = d$$

$$\lambda = \mathbf{d} - \mathbf{p}.\hat{\mathbf{n}}$$

Distance = $\lambda = \dfrac{\mathbf{d} - \mathbf{p}.\mathbf{n}}{|\mathbf{n}|}$

Vector geometry in FM

Find the distance from a point P to a plane **r**.**n** = *d*

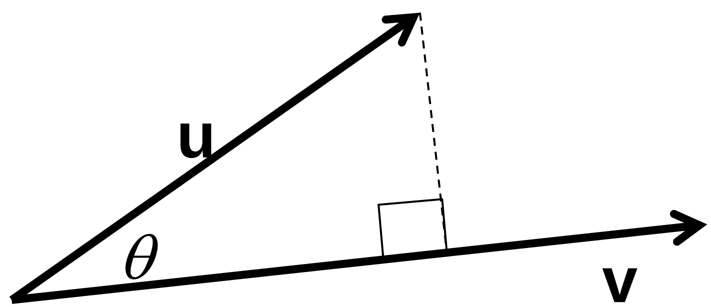
What steps are involved? (Wolfram)

1. Find normal vector **n**
2. Line through P with direction vector **n**
3. Random point Q on plane – vector PQ
4. Projection of PQ onto **n** – length of projection

Vector geometry in FM

What is a projection?

The projection of **u** onto **v** is the resolved part of **u** in the direction of **v**



$$\left(|\mathbf{u}|\cos\theta\right)\hat{\mathbf{v}}$$

$$=(\mathbf{u}.\hat{\mathbf{v}})\hat{\mathbf{v}} \quad =\left(\mathbf{u}.\frac{\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}$$

Vector geometry in FM

Find the distance from a point P to a plane **r**.**n** = *d*

What steps are involved? (Wolfram)

1. Find normal vector **n**
2. ~~Line through P with direction vector **n**~~
3. Random point Q on plane – vector PQ
4. Projection of PQ onto **n** – length of projection

Distance = $\left|(\mathbf{q}-\mathbf{p}).\hat{\mathbf{n}}\right| = \dfrac{\left|\mathbf{q}.\mathbf{n}-\mathbf{p}.\mathbf{n}\right|}{\left|\mathbf{n}\right|} = \dfrac{\left|d-\mathbf{p}.\mathbf{n}\right|}{\left|\mathbf{n}\right|}$

Vector geometry in FM

Problem 3

Finding the distance between skew lines

https://www.geogebra.org/m/dva6mkw6

Vector geometry in FM

Find the distance between the skew lines

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} -14 \\ 3 \\ 14 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

What steps are involved?

1. Write down vector PQ in terms of λ and μ
2. PQ is perpendicular to both lines – find two equations and hence find λ and μ
3. Find the distance PQ

Vector geometry in FM

Find the distance between the skew lines
$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$

What steps are involved (Wolfram)

1. Directions of both lines $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$
2. Vector product $\hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2$
3. Vector between a point on each line $\mathbf{a}_1 - \mathbf{a}_2$
4. Projection of $\mathbf{a}_1 - \mathbf{a}_2$ onto $\hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2$ gives $(\mathbf{a}_1 - \mathbf{a}_2).(\hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2)$
5. Length of this is $|(\mathbf{a}_1 - \mathbf{a}_2).(\hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2)|$ or $\dfrac{|(\mathbf{a}_1 - \mathbf{a}_2).(\mathbf{d}_1 \times \mathbf{d}_2)|}{|(\mathbf{d}_1 \times \mathbf{d}_2)|}$

Vector geometry in FM

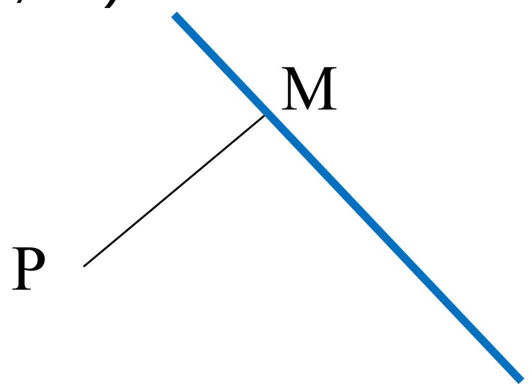
Problem 4

Finding the distance of a point from a line

https://www.geogebra.org/m/t3xkg9xr

Vector geometry in FM

Find the distance from the point P (1, 2, -1) to the line $\mathbf{r} = \begin{pmatrix} 10 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$

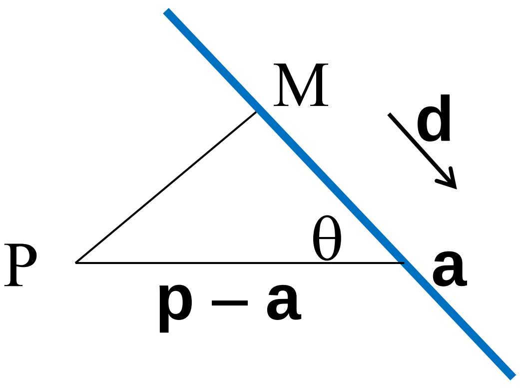


What steps are involved?

1. Write down vector PM in terms of λ
2. PM is perpendicular to the line – find λ
3. Find the distance PM

Vector geometry in FM

Find the distance from the point P (1, 2, -1) to the line $\mathbf{r} = \begin{pmatrix} 10 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$



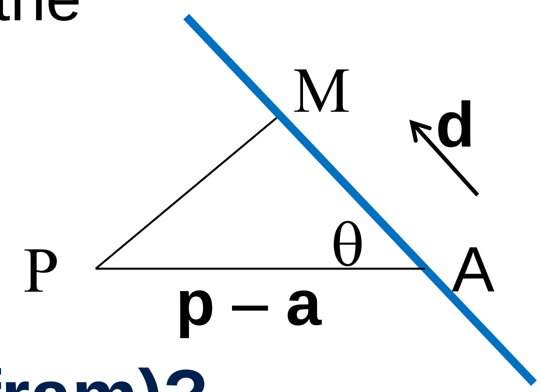
Using the vector product

$$|\mathrm{PM}| = |\mathbf{p} - \mathbf{a}| \sin\theta$$

$$= \left|(\mathbf{p} - \mathbf{a}) \times \hat{\mathbf{d}}\right| = \frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$$

Vector geometry in FM

Find the distance from the point P to the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$

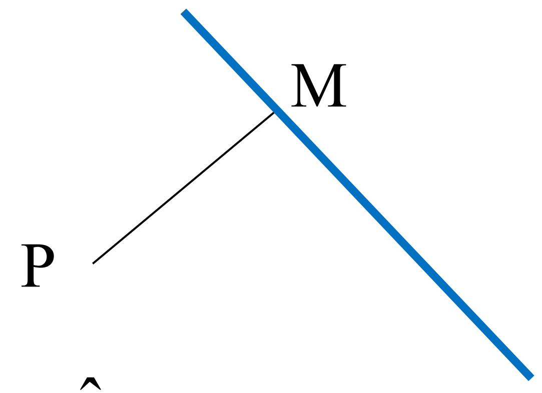


What steps are involved – (Wolfram)?

1. Random point on line $\mathbf{a}$, direction of line $\hat{\mathbf{d}}$
2. Find projection of AP onto $\hat{\mathbf{d}}$
   i.e. $((\mathbf{p} - \mathbf{a}).\hat{\mathbf{d}})\,\hat{\mathbf{d}}$
3. PM = PA + AM = $(\mathbf{a} - \mathbf{p}) + ((\mathbf{p} - \mathbf{a}).\hat{\mathbf{d}})\,\hat{\mathbf{d}}$

Vector geometry in FM

Find the distance from the point P to the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$



What steps are involved?

1. Write down vector PM in terms of λ
2. PM is perpendicular to the line – find λ
3. Find the distance PM

$$\mathbf{a} + \lambda\hat{\mathbf{d}} - \mathbf{p}$$

$$\left(\mathbf{a} - \mathbf{p} + \lambda\hat{\mathbf{d}}\right).\hat{\mathbf{d}} = 0$$

$$\lambda = (\mathbf{p} - \mathbf{a}).\hat{\mathbf{d}}$$

$$\mathbf{a} - \mathbf{p} + \left((\mathbf{p} - \mathbf{a}).\hat{\mathbf{d}}\right)\hat{\mathbf{d}}$$

About MEI

- Registered charity committed to improving mathematics education
- Independent UK curriculum development body
- We offer continuing professional development courses, provide specialist tuition for students and work with employers to enhance mathematical skills in the workplace
- We also pioneer the development of innovative teaching and learning resources