Pythagoras, and the Napkin Ring

C.D. Wright
maths@solipsys.co.uk

29th June, 2018

Abstract

The "Napkin Ring Problem" is well known as a puzzle, but the ideas and concepts associated with it are wide and deep.

At the MEI conference, 2018, these ideas were presented in a session - this is a sketch of those concepts and the connections between them. This document points the way to the ideas, but does not contain the details.

1 Pythagoras and the Napkin Ring

The story begins decades ago, when I was in a gathering of people, and someone said "So, you like puzzles?" "Yes," I replied.

"And you study maths?" Again, "Yes," I replied.

"Well, here's a puzzle you can solve and give the answer to as a simple, rational multiple of pi. No complex formulas, and you can probably do the calculations in your head - are you ready?"

And so I was introduced to what I now know to be the Napkin Ring Problem. As you might imagine there's more to the story, but I was set the puzzle like this:

> We start with a sphere, and through it we drill a circular hole. The length of the hole is 6 inches.
>
> What volume remains?

As luck would have it the problem falls quickly to a classic problem solving technique, but the point is that the result feels like, well, "Magic", and as such it feels somewhat unsatisfactory. So while I did, in fact, solve the puzzle, much to the annoyance of the person challenging me, I wanted to know more.

In exploring the problem and associated concepts I then found that there is a lot going on. ***Really, a lot.***

I started to chase down the ideas and the connections between them. Not all are presented here, and certainly they aren't presented in depth, nor are all the details given.

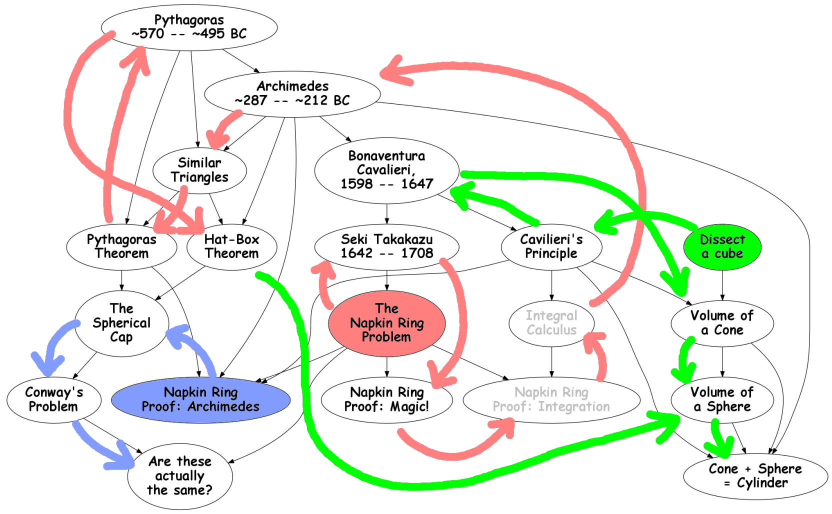

Figure 1: The Web of Concepts, and our paths through it

So this is a romp through the ideas. When presented at the MEI we took three paths through them, and they've been colour-coded here as Red, Green, and Blue.

We'll take the paths in turn, starting with the Red Path.

2 Phase 1 : The Red Path

The puzzle was apparently first discussed by the 17th-century Japanese mathematician Seki Kōwa, and you can read about it on the Wikipedia page referenced at the end. But don't do that yet, let's think about it a little first.

The thing is, it feels impossible, it feels like we haven't been given enough information. We know the length of the hole, but we don't know the size of the sphere we start with. And even if we did know the size of the sphere, surely we would then have to do some complex calculations.

But we've been told that we ***don't*** have to do any such complex calculations, and we've been told that we ***can*** solve it, and so that implies that, since we don't know the initial size of the sphere, the answer must be independent of it.

So we can choose the size that makes the calculation easy. Choose an initial sphere the size of the Earth. To get a hole that's just 6 inches long we have to remove effectively ***all*** of the Earth, and leave just a 6 inch wide thin sliver around the Equator.

Does that help?

...

No.

OK, then let's make the initial sphere as small as possible. It has to have a diameter that's at least 6 inches, otherwise there would be no way we could get a 6 inch hole. But if the diameter is exactly 6 inches, then the hole is of zero width, and so the volume that remains is everything. Thus the radius is 3 inches, so the answer is 36π cubic inches.

But this assumes that the answer is independent of the initial radius, and that feels implausible. How can we check that? We can, of course, perform the appropriate integration of the volume of revolution, and that's what I did, but perhaps there is an easier way to see the answer. Perhaps we can do this with the tools that, say, Archimedes has at his disposal. He did clever stuff with volumes of spheres, etc, perhaps we can do the same.

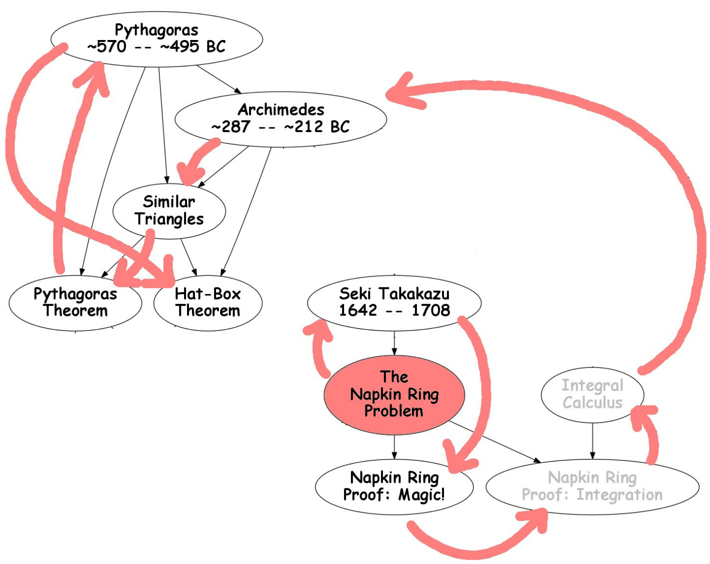

Figure 2: The Red Path

So we warm up by looking at a proof of Pythagoras' Theorem using similar triangles. You can see the proof as Paul Taylor's entry for Round 1 of the Big Internet Math Off - link at the end.

We also use similar triangles in the proof of the Archimedes "Hat-Box Theorem", which says this.

> Consider a sphere, and enclose it in a cylinder. Take two planes parallel to the equator. The surface area of the sphere between the planes is equal to the surface area of the portion of the cylinder between the planes.

The statement along with illustrations can be found here:

http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html

Again, this is easily proved with calculus, but also provable with the techniques Archimedes had to hand.

So in particular, the Hat-Box Theorem, provable using techniques known to Archimedes, says that the surface area of a sphere of radius R is the same as the area of the curved surface of the enclosing cylinder, which is $4\pi R^2$.

---

3 Phase 2 : The Green Path

Now let's change the subject entirely. Take a sheet of A4 and cut out the shape shown here. Fold along the centre line, then fold on that long diagonal. The other folds are at 45 degrees, so the trangle on the left is right-isosceles triangle.

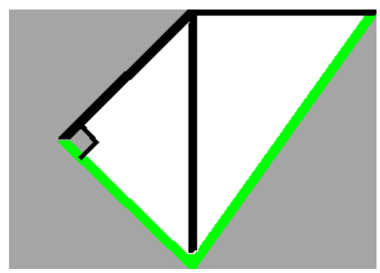

Figure 3: A "thing" to cut out

Take two of these, lay them one on top of the other, and tape them together along the green edges. The resulting "thing" can then be "popped open" to make a skew square-based pyramid. In turn, three of the resulting square-based pyramids can be fitted together to make a cube.

Try it.

So the thing you have is one third of a cube.

Now we can apply Cavilieri's Principle. Take a pile of coins. It has the same volume whether it's stacked "straight", or stacked "curvy". So we can think of our skew square-based pyramid as being made of horizontal slices, and imagine sliding them around to make it into a right square-based pyramid. Similarly, the volume will remain unchanged if we replace each slice with a slice of a different shape but the same area.

Finally, consider replacing each slice with a scaled slice. Putting all this together, *and with some work,* we have shown this:

> Take any planar shape and a point outside the plane. Join every point on the perimeter to the given point, making a pyramid or cone. The volume of the resulting shape is one third times the height times the area of the base.

And now, consider a solid sphere, and a small patch on its surface. Join that small patch to the centre, making a cone. The volume of that cone is 1/3 times the radius times the area of the patch.

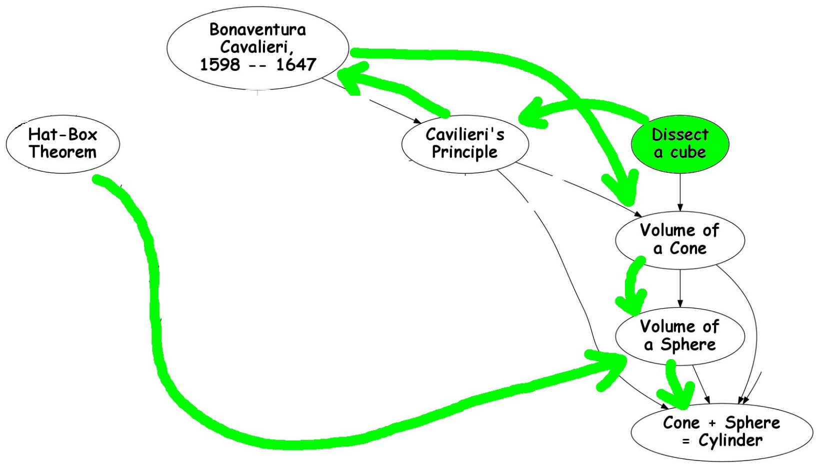

Figure 4: The Green Path

Now cover the entire surface with small, non-overlapping patches. The sum of the volumes of the cones is 1/3 times the radius times the sum of the areas of the patches. But the sum of the areas of the patches is the entire surface, and is thus $4\pi R^2$. So we have:

The volume of a sphere is $\frac{1}{3}(4\pi R^2)R$.

Which is $\frac{4}{3}\pi R^3$.

---

4 Phase 3 : The Blue Path

And now we are ready to tackle the Napkin Ring Problem using just the techniques that Archimedes had. Take the Napkin Ring, and place beside it a sphere with diameter the same as the length of the hole.

Take a single slice at an arbitrary height. The idea is that if we can show that the area of the slice through the Napkin Ring, which is an annulus, is the same as the slice through the sphere, which is a circle, then the volumes must be the same.

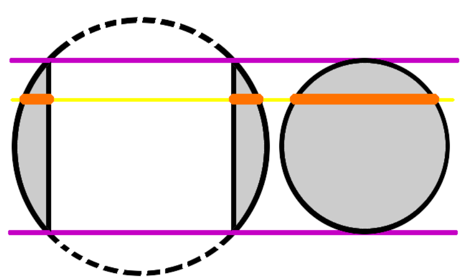

Figure 5: Slicing the Napkin Ring

And now that's your challenge. Can you show that the areas are equal? It's easy when you find it, but until you do, it seems impossible. And that's like so much of mathematics. It's easy when you see the right technique, the right diagram, the right calculation. But until you do, it's quite simply impossible.

That's why we need to do lots and lots of practice. It's only with practice that we can develop the intuition that makes it easier to choose the right approach.

So by pulling together all these ideas we can prove that the volume of the Napkin Ring truly is independent of the size of the original sphere, a result that seems quite amazing.

But you'll notice there are further "blobs" on the blue path.

We return to surface areas, and consider the following problem.

Take a sphere, and from the "north pole" drill a straight hole in any direction you like through the sphere until it emerges somewhere. The point of emergence defines a line of latitude.

Consider the surface area of the spherical cap above that line if latitude.

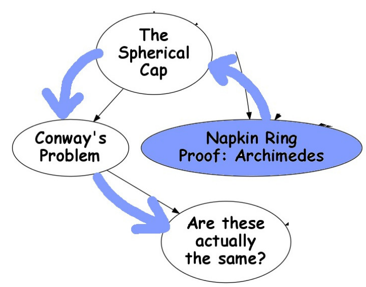

Figure 6: The Blue Path

> Claim: If the length of the hole is L, the area of the spherical cap is πL^2.

You have the tools to tackle this, it just requires Pythagoras and the Hat-Box Theorem. But while that's an interesting little challenge, it's not the real question. The real question is this:

> The volume of the Napkin Ring is independent of the size of the original sphere. The area of the spherical cap is independent of the original spherical.
>
> Are these, in some sense, the same question?

I don't think so, but I'm willing to be convinced otherwise!

---

5 References

- https://en.wikipedia.org/wiki/Napkin_ring_problem
- https://aperiodical.com/2018/07/the-big-internet-math-off-round-1-samuel-hansen-v-paul-taylor/
- http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html
- https://en.wikipedia.org/wiki/Cavalieri
- https://www.solipsys.co.uk/new/ColinsBlog.html