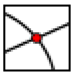
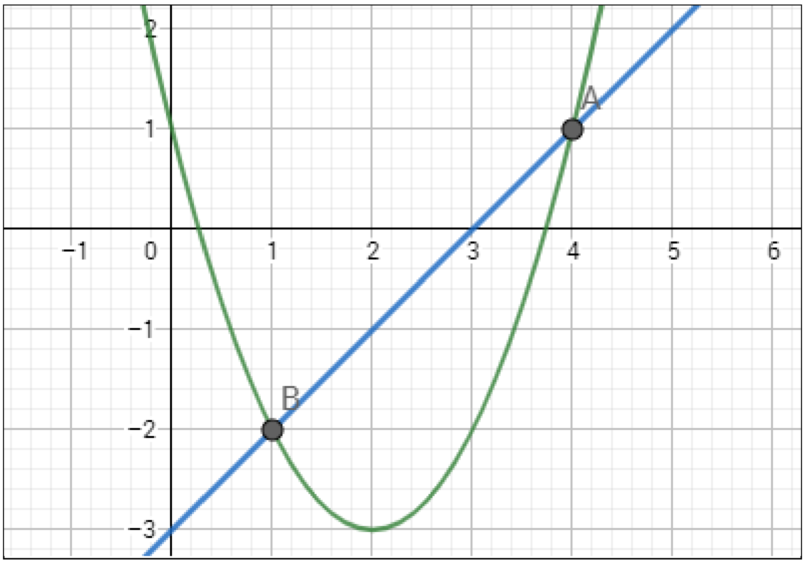
MEI GeoGebra Tasks for AS Pure

Task 1: Coordinate Geometry – Intersection of a line and a curve

1. Add a quadratic curve, e.g. **y = x² – 4x + 1**
2. Add a line, e.g. **y = x – 3**
3. Use the **Intersect** tool  to find the points of intersection of the line and the curve.



Questions for discussion

- What is the relationship between the *x*-coordinates of the points of intersection and the equations of the line and curve?
- Does this work for other curves and lines?

Problem *(Try the problems with pen and paper first then check it on your software)*

Find exact values of the coordinates of the points of intersection of the following:

$y = x^2$ and $y = 2x + 3$ $\qquad$ $y = x^2 - x$ and $y = 2 - x$ $\qquad$ $y = x^2 - 2x + 2$ and $y = 2x + 1$

Further Tasks

- Can you find an example of a line and a curve that would have:
  - Exactly 1 point of intersection?
  - No points of intersection?
- Investigate the number of points of intersection of two curves.
- Investigate the intersection of a line and a circle.

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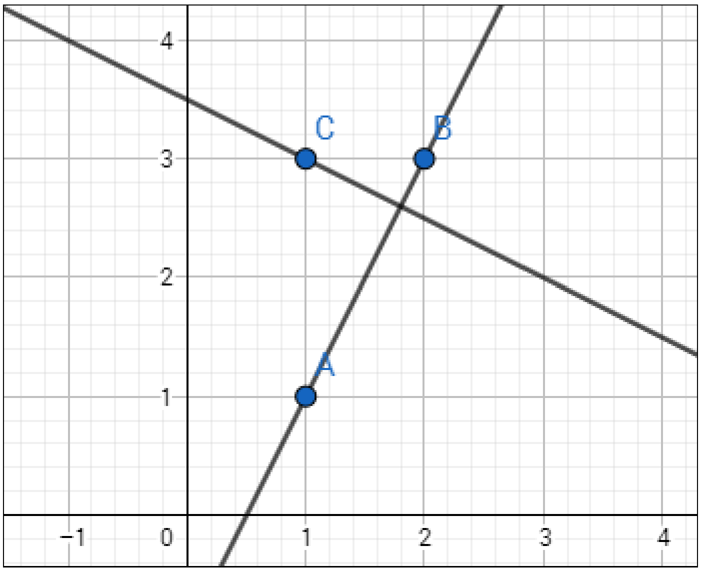
Task 2: Coordinate Geometry – Perpendicular lines

1. Use the **Point** tool to add points, **A** and **B**.
2. Use the **Line** tool to create the line through **A** and **B**.
3. Use the **Point** tool to add a point, **C**.

Click on the point C and then the line.

4. Use the **Perpendicular Line** tool (in *Construct* tools) to create the line through **C** and perpendicular to the line **AB**.



Questions for discussion

- What is the relationship between the equations of the lines?
- What is the relationship between the equations of the lines then they are written in the form $y = mx + c$?

Problem *(Try the problem with pen and paper first then check it on your software)*

Show that the line perpendicular to the line through $(5,1)$ and $(1,3)$ that passes through the point $(3,4)$ has equation $y = 2x - 2$.

Further Tasks

- For two points A and B what are the possible positions for C so that the line through C is a perpendicular bisector?
- For three points A, B and C find the point of intersection of the two lines.

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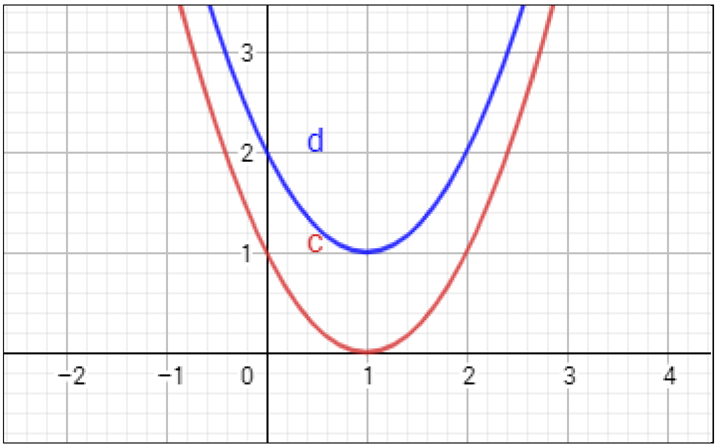
Task 3: Algebra – Graphs of quadratic functions

1. Plot the curve: **y = (x + a) (x + b)**
   If prompted click *Create Sliders*.

2. Plot the curve: **y = (x + p)^2 + q**
   If prompted click *Create Sliders*.



Questions for discussion

- Can you find values for *a*, *b*, *p* and *q* so that the two graphs are the same?

- What is the relationship between the values of *a*, *b*, *p* and *q* when the graphs are the same?

Problem *(Try the problems with pen and paper first then check it on your software)*

Solve the equation $x^2 - 2x - 8 = 0$ by both factorising and completing the square.

Further Tasks

Change the equation in step 2 to **y = k (x + p)^2 + q** (with a slider for *k*).

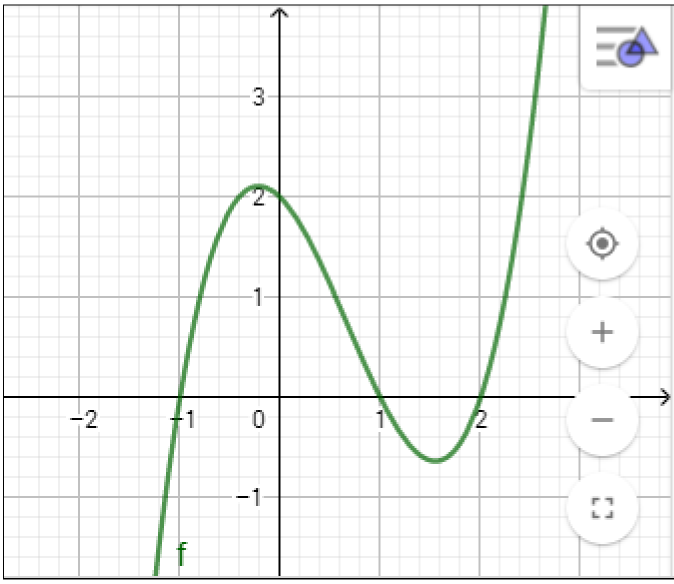
- Where does this curve cross the *x*-axis?
- Can you change the equation in step 1 so the curves are the same?

MEI GeoGebra Tasks for AS Pure

Task 4: Algebra – The Factor Theorem

GeoGebra Classic version

1. Plot the function **f(x) = x^3 – $2x^2$ – x + 2**
2. Enable the spreadsheet: **View > Spreadsheet**
3. In column A enter the values –3, –2, –1, 0 , 1, 2, 3, 4
4. In Cell B1 enter **=f(A1)** and fill-down.



	A	B
1	-3	-40
2	-2	-12
3	-1	0
4	0	2
5	1	0
6	2	0
7	3	8
8	4	30
9		
10		
11		

Questions for discussion

- How do this table and graph confirm that $x^3 - 2x^2 - x + 2 = (x+1)(x-1)(x-2)$?
- Can you find the factors of the following cubics:
  $y = x^3 + 4x^2 + x - 6$ $\qquad$ $y = x^3 - 4x^2 - 11x + 30$
  $y = x^3 - x^2 - 8x + 12$ $\qquad$ $y = x^3 - 7x^2 + 36$

Problem (*Try the question with pen and paper first then check it on your software)*

Show that $(x-2)$ is a factor of $\mathrm{f}(x) = x^3 + 4x^2 - 3x - 18$. Hence find all the factors of $\mathrm{f}(x)$.

Further Tasks

- Find examples of cubics that only have one real root.
- Investigate using the factor theorem for polynomials of other degrees, e.g. quadratics or quartics.
- Investigate the **Factor** and **Expand** commands in the CAS view.

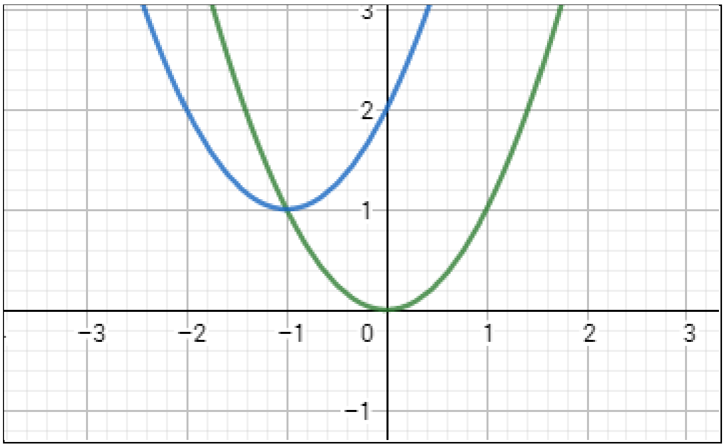
MEI GeoGebra Tasks for AS Pure

Task 5: Functions – Transformations

1. In the Input bar enter: **f(x) = x²**

   It is essential that this is entered as a function f(x).

2. In the Input bar enter: **g(x) = f(x + a) + b**
   If prompted click to create sliders for **a** and **b**.



Questions for discussion

- What transformation maps f(x) onto g(x)?
- Does this work if other functions are entered for f(x)?

Problem *(Try the problem with pen and paper first then check it on your software)*

Show that $(x+2)^3 + 3 = x^3 + 6x^2 + 12x + 11$.
Hence sketch the graph of $y = x^3 + 6x^2 + 12x + 11$.

Further Tasks

- Show that $\mathrm{f}(x) = x^4 - 8x^3 + 24x^2 - 32x + 13$ can be written in the form $(x+a)^4 + \mathrm{b}$ and hence find the coordinates of the minimum point on the graph of $y = \mathrm{f}(x)$.
- Create sliders for **c** and **d**.
  In the Input bar enter: **h(x)=c*f(d*x)**.
  What transformation maps f(x) onto h(x)?

  Changing f(x) to f(x)=x^3–x might help make it clearer.

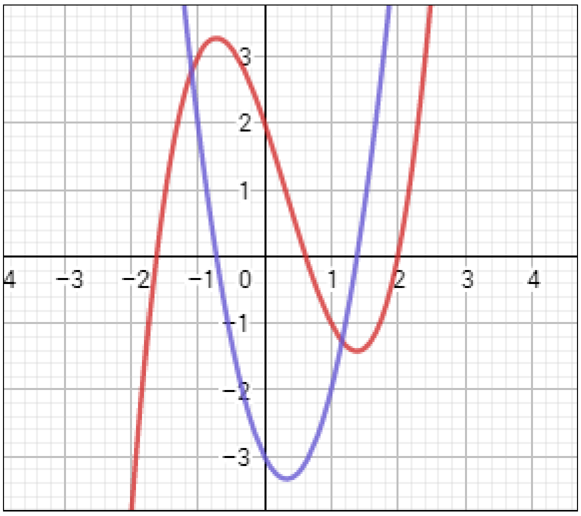
MEI GeoGebra Tasks for AS Pure

Task 6: Differentiation – Exploring the gradient on a curve

Use $\frac{d}{dx}$ from the *f(x)* keyboard or type **f ′(x)**

1. Plot a cubic function: e.g. $\mathbf{y = x^3 - x^2 - 3x + 2}$

2. Plot the gradient function by entering **Derivative(f)** in the input bar.

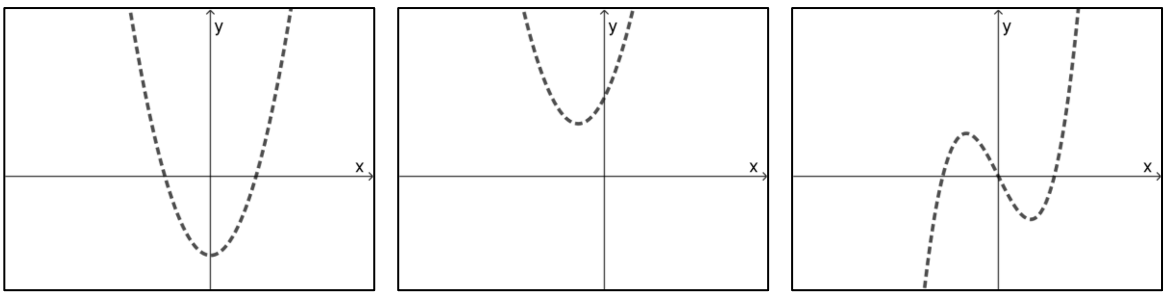


Question for discussion

- How is the shape of the gradient graph related to the shape of the original graph? Verify your comments by trying some other functions for f(x).

Problem

Change your original function in GeoGebra so that its gradient function is one of the following:



Extension Task

Find the point on the function $f(x) = x^3 - 6x^2 + 9x - 1$ where the tangent has its maximum downwards slope. Investigate the point with maximum downward slope for other cubic functions.

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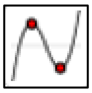
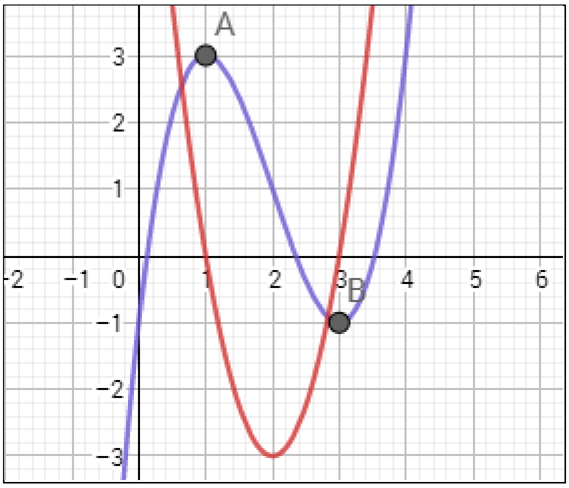
Task 7: Differentiation – Stationary points

1. Plot a cubic function: e.g. **f(x) = x³ − 6x² + 9x – 1**
2. Use the **Turning Point** (or **Extremum)** tool  to find the turning points of the function.
3. Plot the gradient function by entering **Derivative(f)** in the input bar.

Use $\frac{d}{dx}$ from the *f(x)* keyboard or type **f ′(x)**



Question for discussion

- How can you use the graph of the gradient function to explain why the function has a local maximum at A and a local minimum at B?

Verify your comments by trying some other functions for f(x).

Problem *(Try the problem with pen and paper first then check it on your software)*

For the following curves plot the graphs and their derivatives. Use the derivative graph to find where the curve has a maximum or minimum:

$y = x^2 + 4x + 1$ $\qquad$ $y = 4 - 6x - x^2$ $\qquad$ $y = x^3 - 3x$ $\qquad$ $y = x^3 - 3x^2 + 3x$

Extension Task

Use GeoGebra to find the gradient function of $\mathrm{f}(x) = x^3 - 6x^2 + 12x - 5$. Explain why the function has a stationary point that is neither a maximum nor a minimum (*a stationary point of inflection*).

Find some other functions that have stationary points of inflection.

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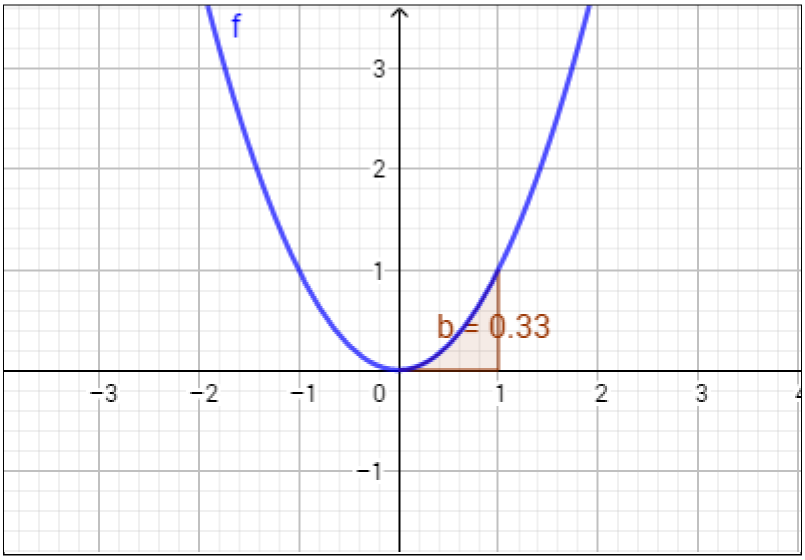
Task 8: Integration – Area under a curve

1. In the Input bar enter: **f(x)=x²**

It is essential that this is entered as a function f(x).

2. In the Input bar enter: **A = Integral(f, 0, a)**
If prompted click *Create Sliders*.

Use ∫ from the *f(x) keyboard* or type **Integral**



Questions for discussion

- What is the relationship between the area and the value of *a*?
- What is the relationship if f(*x*) is changed to a different power of *x*?

Problem *(Try the problem with pen and paper first then check it on your software)*

Find the area under $f(x) = x^5$ between $x = 0$ and $x = 3$.

Further Tasks

- Investigate the area under $f(x)= x^n$ between $x = a$ and $x = b$.
- Investigate the areas under functions that are the sums of powers of *x*:
e.g. $f(x)=x^3 + 3x^2 + 4x +1$

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Task 9: Trigonometry – Trigonometric equations

1. Plot the function: **y = sin(x°)**

> The degrees sign can be found in the f(x) menu in the onscreen keyboard or α menu in the input bar in Classic 5

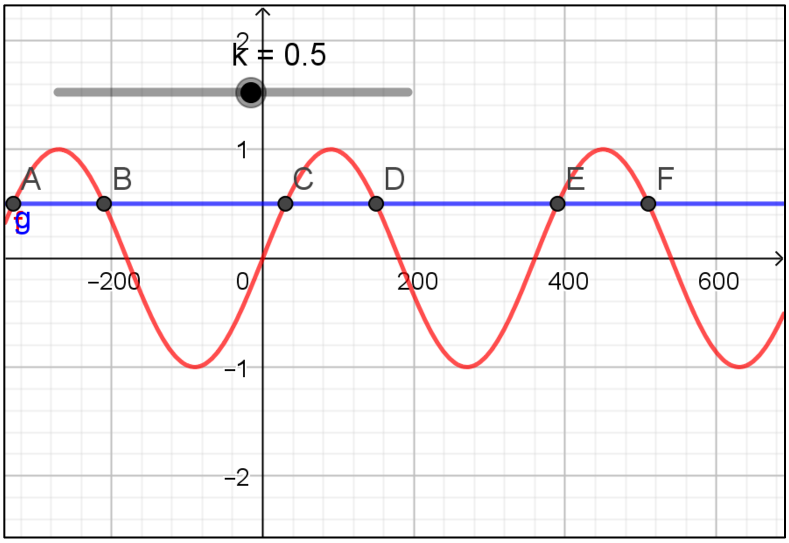
2. Set the x-axis to have a range -360 < x < 720

   In the app use from the settings menu.

   In Classic use **Move Graphics View** and drag the *x*-axis.

3. Enter the function: **y = k**
   If prompted click *Create Sliders*.

4. In the input bar enter: **Intersect(f, g, -360, 720)**



Questions for discussion

- What symmetries are there in the positions of the points of intersection?
- How can you use these symmetries to find the other solutions based on the value of $\sin^{-1}x$ given by your calculator? (This is known as the "principal value".)

Problem (*Try the question just using the sin^{-1} function on your calculator first then check it using the software)*

Solve the equation: $\sin x = 0.2$ $(-360° \leq x \leq 720°)$

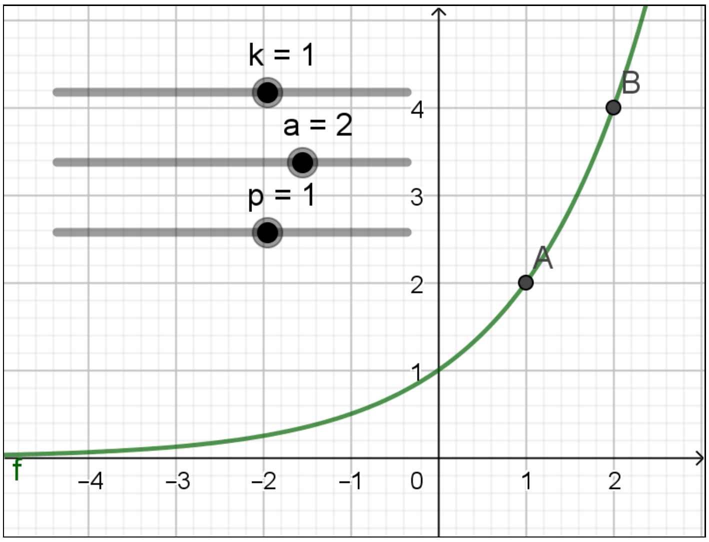
Further Tasks

- Investigate the symmetries of the solutions to $\cos x = k$ and $\tan x = k$.
- Investigate the symmetries of the solutions to $\sin 2x = k$.

MEI GeoGebra Tasks for AS Pure

Task 10: Exponentials and logarithms – Graph of $y = ka^x$

1. Enter the function: **y = k×a$^{\mathbf{x}}$**
   If prompted click *Create Sliders*.
2. Add the point: **(p, f(p))**
3. Add the point: **(p+1, f(p+1))**



Questions for discussion

- How does varying **k** affect the curve?
- How does varying **a** affect the curve? Why is it sensible to restrict **a** to positive values?
- What is the relationship between the *y*-coordinates of the points **A** and **B**?

Problem *(Try the problem with pen and paper first then check it on your software)*

The function $y = ka^x$ passes through the points $(1,10)$ and $(3,160)$.
Find the values of a and k.

Further Tasks

- Investigate solving $ka^x = b$ graphically and by using logs.
- Use the graph of $y = a^x$ to explain why $a^{m+n} = a^m a^n$.

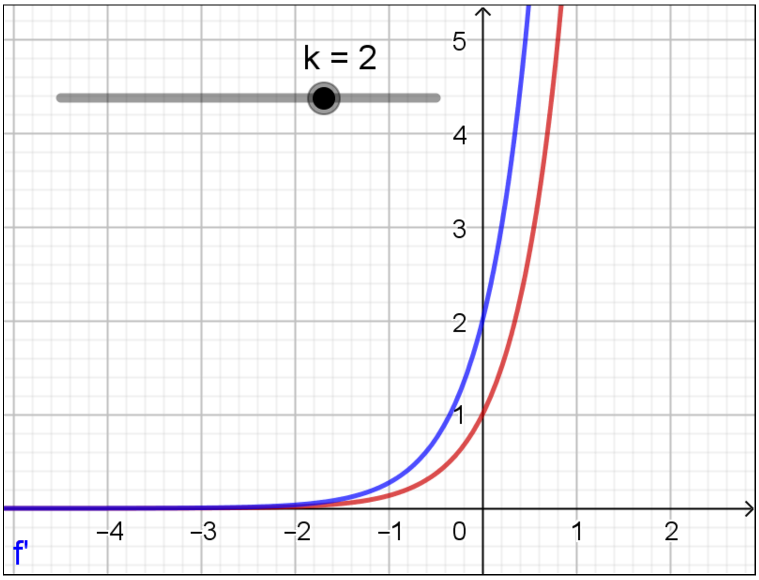
MEI GeoGebra Tasks for AS Pure

Task 11: Derivate of exponential functions y = e^{kx}

1. Plot the curve: **y = e$^{\mathbf{kx}}$**
   If prompted click *Create Sliders*.

Use $\frac{d}{dx}$ from the *f(x)* keyboard or type **f ′(x)**

2. Plot the gradient function by entering **Derivative(f)** in the input bar.



Question for discussion

- How can you express the gradient function for $y = \mathrm{e}^{kx}$?

Problem (*Check your answer by plotting the graph and the tangent on your software)*

Find the equation of the tangent to the curve $y = \mathrm{e}^{2x}$ at the point $x = 1$.

Further Tasks

- Find the tangent to $y = \mathrm{e}^{x}$ that passes through the origin.
- Find the gradient of the tangent to $y = 3^{x}$ when $x = 0$.

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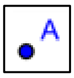
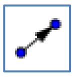
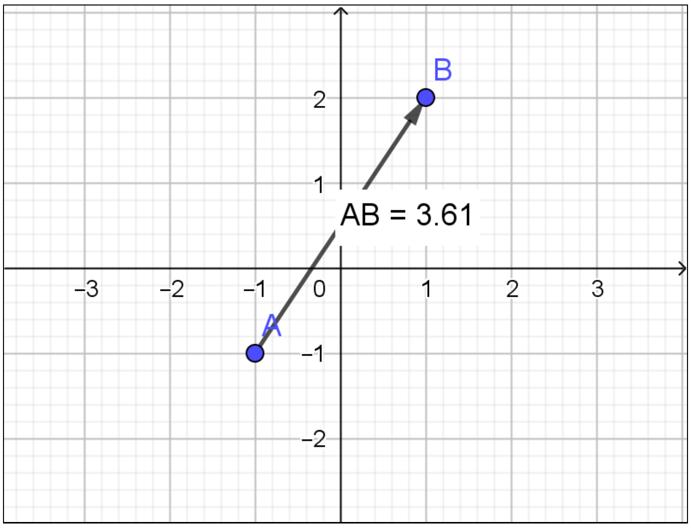
Task 12: Introduction to vectors

1. Use the **Point** tool  to add points, **A** and **B**.

2. Use the **Vector** tool  to create the vector AB.

3. Use the **Distance or Length** tool  measure the length AB.

Click on the point A then point B.



Questions for discussion

- What is the relationship between the vector $\overrightarrow{AB}$ and length $|\overrightarrow{AB}|$?

- Can the length $|\overrightarrow{AB}|$ ever be negative?

Problem *(Try the problem with pen and paper first then check it on your software)*

The vector $\overrightarrow{AB} = \begin{pmatrix} 3 \\ k \end{pmatrix}$ has length $|\overrightarrow{AB}| = \sqrt{34}$. Find the possible values of k.

Further Tasks

- On a new page add two position vectors such as **u = (1,2)** and **v = (-2,3)**. Investigate the vectors **u + v**, **u – v** and **2u**.

- Use the 3D view/app to investigate 3D vectors.

Using a lower case letter creates a vector instead of a point.

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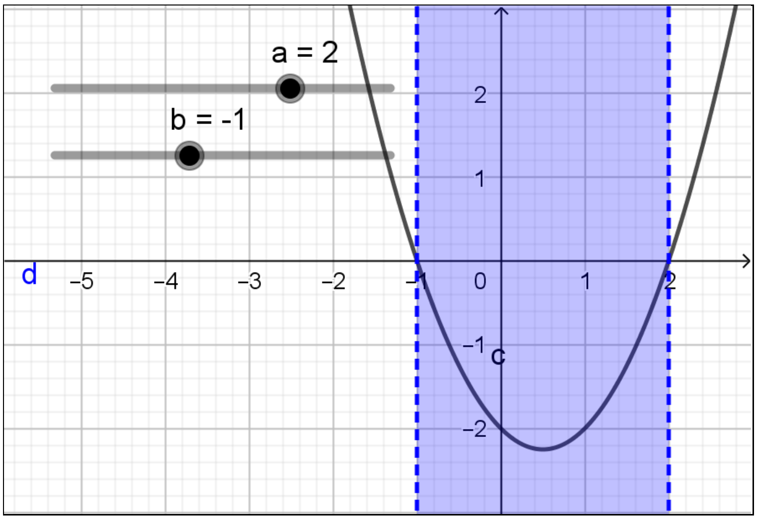
Task 13: Quadratic Inequalities

1. Plot the curve: **y=(x-a)(x-b)**
   If prompted click *Create Sliders*.

2. Plot the inequality: **(x-a)(x-b)<0**



Questions for discussion

- If the product of two numbers is negative what does this tell you about the numbers?
- Will you always be able to find x-values for which a quadratic is negative?
- What would the solution to $(x-a)(x-b) > 0$ look like?

Problem *(Try the problem with pen and paper first then check it on your software)*

Sketch the graph of $y = 2x^2 - x - 6$ and hence solve the inequality $2x^2 - x - 6 \geq 0$.

Further Tasks

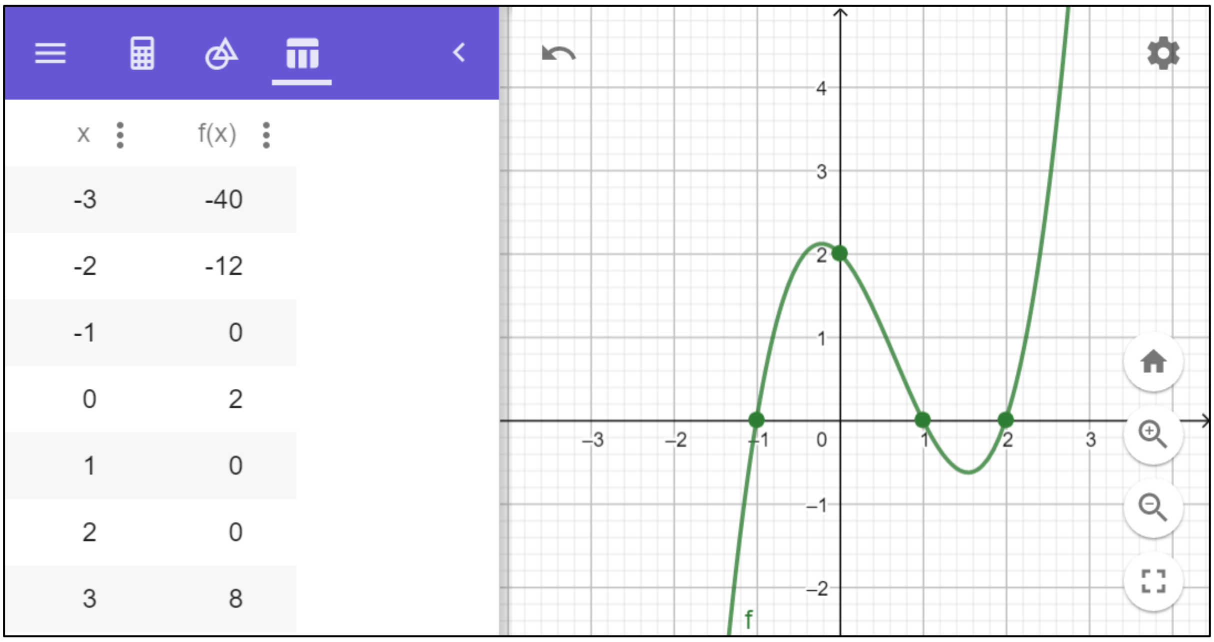
- Find the range of values for k such that $x^2 - 4x + 3 = kx$ has two distinct roots.
- Investigate $y > mx + c$ and $y > ax^2 + bx + c$ graphically.

MEI GeoGebra Tasks for AS Pure

Task 14: Algebra – The Factor Theorem

GeoGebra Graphing Calculator version

1. Plot the function **$y = x^3 - 2x^2 - x + 2$**
2. Click on the 3 dots next to the function in the Algebra view and select *Table of values*
3. Set the start value for x to –4, the end value to 4 and click OK



Questions for discussion

- How do this table and graph confirm that $x^3 - 2x^2 - x + 2 = (x + 1)(x - 1)(x - 2)$?
- How can you use a table and/or a graph to find the factors of the following cubics:
  $y = x^3 + 4x^2 + x - 6$ $\qquad$ $y = x^3 - 4x^2 - 11x + 30$
  $y = x^3 - x^2 - 8x + 12$ $\qquad$ $y = x^3 - 7x^2 + 36$

Problem (*Try the question with pen and paper first then check it on your software)*

Show that $(x - 2)$ is a factor of $f(x) = x^3 + 4x^2 - 3x - 18$. Hence find all the factors of $f(x)$.

Further Tasks

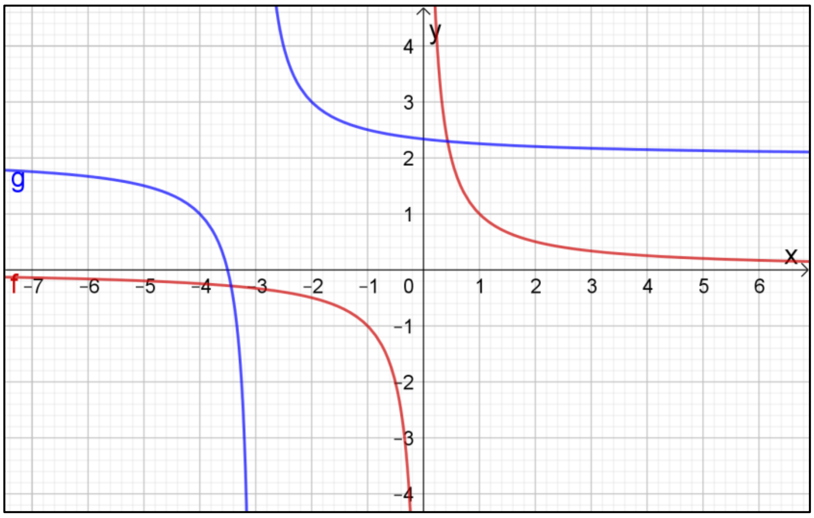
- Find examples of cubics that only have one real root.
- Investigate using the factor theorem for polynomials of other degrees, e.g. quadratics or quartics.

MEI GeoGebra Tasks for AS Pure

Task 15: Functions – Transformations of $y = \frac{k}{x}$ curves

1. In the Input bar enter: **y = k/x**
   The software should define this function as f.

2. In the Input bar enter: **y = f(x + a) + b**
   The software should define this function as g.



Questions for discussion

- What are the equations of the asymptotes on the original curve and the transformed curve?
- How does changing k affect the curves?

Problem *(Try the problem with pen and paper first then check it on your software)*

Sketch the graph of $y = \frac{2}{x-1} + 3$.
State the equations of the asymptotes and the points of intersection with the axes.

Further Tasks

- Investigate curves of the form $y = \frac{k}{x^2}$ and $y = \frac{k}{x^2+a} + b$.
- Investigate the points of intersection of the curve $y = \frac{1}{x}$ and the line $y = mx + c$.

MEI GeoGebra Tasks for AS Pure

Teacher guidance

Using these tasks

These tasks are designed to help students in understanding mathematical relationships better through exploring dynamic constructions. They can be accessed using the computer-based version of GeoGebra or the tablet/smartphone app. Each task instruction sheet is reproducible on a single piece of paper and they are designed to be an activity for a single lesson or a single homework task (approximately).

The tasks have been designed with the following structure –

- **Construction:** step-by-step guidance of how to construct the objects in GeoGebra. Students will benefit from learning the rigorous steps need to construct objects and this also removes the need to make prepared files available to them. If students become confident with using GeoGebra they can be encouraged to add additional objects to the construction to aid their exploration.

- **Questions for discussion:** This discussion can either be led as a whole class activity or take place in pairs/small groups. The emphasis is on students being able to observe mathematical relationships by changing objects on their screen. They should try to describe what happens, and explain why.

- **Problem:** Students are expected to try the problem with pen and paper first then check it on their software. The purpose is for them to formalise what they have learnt through exploration and discussion and apply this to a “standard” style question. Students could write-up their answers to the discussion questions and their solution to this problem in their notes to help consolidate their learning and provide evidence of what they’ve achieved. This problem can be supplemented with additional textbook questions at this stage if appropriate.

- **Further Tasks:** Extension activities with less structure for students who have successfully completed the first three sections.

Task 1: Coordinate Geometry – Intersection of a line and a curve

This task can be used to introduce the intersection of a line and a curve.

Students should consider the equation formed by subtracting the linear function from the quadratic and observing its roots. Some students might find it helpful to plot this.

Problem solutions:
$y = x^2$ and $y = 2x + 3$ (–1, 1) and (3, 9)
$y = x^2 - x$ and $y = 2 - x$ (–2, 4) and (1, 1)
$y = x^2 - 2x + 2$ and $y = 2x + 1$ (–√3+2, –2√3+5) and (√3+2, 2√3+5)

Task 2: Coordinate Geometry – Perpendicular lines

The equations of the lines will appear in the form $ax + by = c$. Students should try rewriting these in the form $y = mx + c$ or get the software to do it automatically (right-clicking on the equations).

The equations of the perpendicular lines will have the coefficients written in decimal form and it might be helpful to discuss with students why writing these a fractions is preferable in some circumstances.

Problem solution: The perpendicular line has equation $y = 2x - 2$.

Task 3: Algebra – Graphs of quadratic functions

Students should attempt to solve some quadratic equations by completing the square and then making x the subject. Students might find it helpful to observe the line of symmetry of the curve or the relationship between the completed square solution and solving with the standard formula.

Problem solution: $x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$ $\quad x = 4$ or $x = -2$

$(x-1)^2 - 9 = 0$ $\quad x = 1 \pm 3$ $\quad x = 4$ or $x = -2$

Task 4/14: The Factor Theorem

There are two versions of this task: task 4 has instructions for using GeoGebra Classic and task 14 has instructions for the GeoGebra Graphing Calculator. The mathematical content of both tasks is the same.

This task is intended to reinforce the link between the numerical values of roots, algebraic factors and points of intersection with the *x*-axis. In discussions students should be encouraged to explain how *both* the table and the graph indicate what the factors are.

It might be useful for some students to practise expanding products of three factors before attempting this task.

Students will also need to be shown, or to develop, strategies for dividing by a factor, such as equating coefficients, long division or division by the box method.

Questions:

$y = x^3 + 4x^2 + x - 6$:	$y = (x-1)(x+2)(x+3)$
$y = x^3 - 4x^2 - 11x + 30$	$y = (x-5)(x-2)(x+3)$
$y = x^3 - x^2 - 8x + 12$	$y = (x-2)^2(x+3)$
$y = x^3 - 7x^2 + 36$	$y = (x-6)(x-3)(x+2)$

The third question can be used to demonstrate an example of a cubic with a repeated root.

Problem solution:
$x^3 + 4x^2 - 3x - 18 = (x-2)(x+3)^2$

Task 5: Functions – Transformations

If students have met trigonometric functions then these work well for this task.

For the problem students should expand the function using either a binomial expansion or by multiplying out the brackets.

The graph of the function is the graph of $\mathrm{f}(x) = x^3$ translated by $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

Be careful with second further task (horizontal stretches) – they can look like vertical stretches for many functions but this is an excellent discussion point.

Students should also take care with the scales on the axes here as these can cause confusion. $\mathrm{f}(x)=\sin x$ or $\mathrm{f}(x)=x^3-x$ are good functions to use for this.

Task 6: Differentiation – Exploring the gradient on a curve

The aim of this task is for students to investigate (or verify if they have already met it) the shape of derivative functions. They should be encouraged to discuss why the derivatives have the shape they do in terms of the gradient of the tangent to the curve at different points. It can be used as an introduction to the topic or to consolidate what they have already learnt.

Problem solution (possible solutions):
$\mathrm{f}(x) = x^3 - x$ $\mathrm{f}(x) = x^3 + x^2 + x$ $\mathrm{f}(x) = x^4 - x^2$

Task 7: Differentiation – Introduction to Stationary Points

This task highlights the link between the derivative and determining the nature of stationary points on curves. Students should be encouraged to consider how the derivative crosses the x-axis (+ve to –ve or –ve to +ve) to determine the nature of the stationary points.

Problem solutions:
$y = x^2 + 4x + 1$ min: (–2, –3)
$y = 4 - 6x - x^2$ max: (–3, 13)
$y = x^3 - 3x$ min: (1, –2) max: (–1, 2)
$y = x^3 - 3x^2 + 3x$ no maxima or minima, it has a stationary point of inflection at (1, 1)

The final example can be used to discuss stationary points that are points of inflection and this can lead into the extension tasks.

Task 8: Integration – Area under a curve

The aim of this task is for students to investigate (or verify if they have already met it) the rule for integrating/finding the area under polynomials. It can be used as an introduction to the topic or to consolidate what they've already learnt.

Problem solution: The area is 20.25.

The first of the further tasks is an opportunity for students to investigate:

$$\int_a^b \mathrm{f}(x)\mathrm{d}x = \int_0^b \mathrm{f}(x)\mathrm{d}x - \int_0^a \mathrm{f}(x)\mathrm{d}x .$$

Task 9: Solutions of Trigonometric Equations (Degrees)

This task encourages students to think about the symmetries of the trigonometric graphs and use these in finding solutions to equations.

Students should be familiar with scaling one of the axes independently by either dragging the axis (as in instruction 2) or by setting it via the Graphics properties.

Problem solution:
$x = -348.46°, -191.54°, 11.54°, 168.46°, 371.54°, 528.46°$.

Task 10: Exponentials and logarithms – Graph of $y = k^{ax}$

This task allows students to investigate exponential functions of the form $y = ka^x$. The points A and B encourage students to focus on the growth factor as a multiple when increasing the value of x by 1.

Problem solution: $y = 2.5 \times 4^x$.

Task 11: Derivates of exponential functions y=e^{kx}

This task can be done on its own or with task 7. The aim of this task is for students to be able to find the gradients and equations of tangents to exponential functions.

Students should observed that the derivative is the same as the y-coordinate for $y = \mathrm{e}^x$ before exploring other curves of the form $y = \mathrm{e}^{kx}$.

Problem solution:
$y = 14.778x - 7.389$

The second of the further tasks requires students to rewrite $y = 3^x$ as $y = \mathrm{e}^{(\ln 3)x}$.

Task 12: Introduction to vectors

This task provides a basic introduction to finding the magnitude of a vector using Pythagoras' theorem.

For the extension task the position vectors need to be created with a lower case letter for their definition for the object to be defined as a vector.

Problem solution
$k = 5$ or -5

Task 13: Quadratic Inequalities

This task focusses on solving quadratic inequalities by sketching graphs. Students should be encouraged to relate the roots of the quadratic with the possible values of the factors to determine whether the product is positive or negative. Substituting in some values can help confirm the solution is valid.

The software shades the region that satisfies the inequality as a vertical strip. It is useful to discuss with students when the convention of shading the region satisfied is helpful, as opposed to shading the region outside the inequality. You should also highlight that the strip is equivalent to marking a set of values on the number line.

It is important to highlight where the solution can be written as a single inequality and where it should be written as two separate inequalities.

Problem solution: $x \le -\frac{3}{2}$ or $x \ge 2$

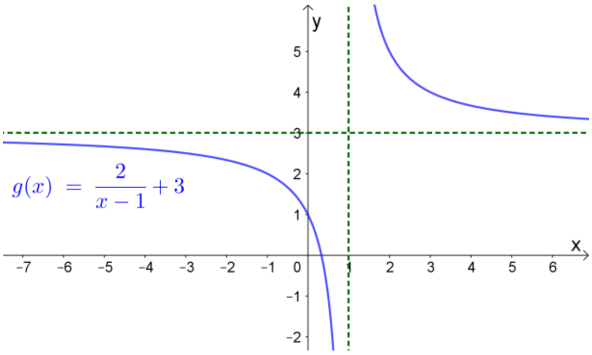
Task 14: The Factor Theorem
See the notes for task 4.

Task 15: Functions – Transformations of $y = \frac{k}{x}$ curves

In this task students are encouraged to explore translations of functions as they relate to $y = \frac{k}{x}$ type curves. The introductory task is useful to remind them about asymptotes and it might be useful to discuss how the equation indicates that there are values of x and y that are not possible as well as how the curve will tend towards these.

For the discussion points the value of k does not change the position of the asymptotes but it does change shapes of the curves.

Problem solution:



Asymptotes: $x = 1$, $y = 3$.
Intersections with the axes: $\left(\frac{1}{3}, 0\right)$, $(0,1)$.