

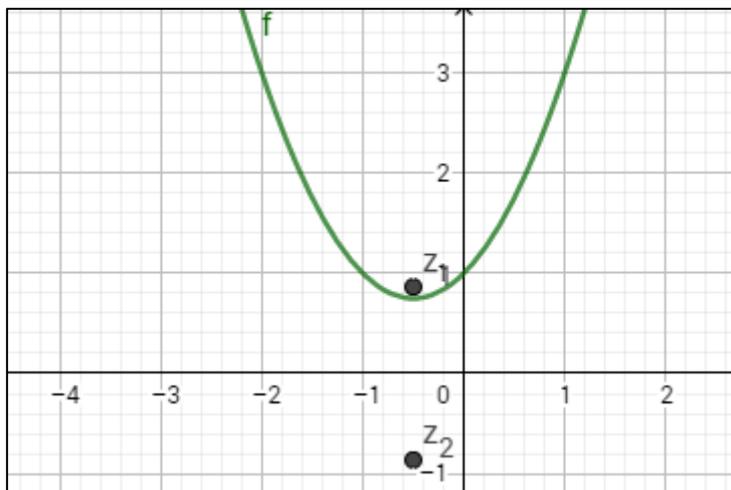
# MEI GeoGebra Tasks for Further Maths: Pure

## Task 1 – Complex numbers: Roots of quadratic equations

1. In the input bar enter the function:  $f(x)=x^2+bx+c$   
If prompted click *Create Sliders*.

It is important that this is created as a function.

2. In the input bar enter the function: **ComplexRoot(f)**



### Questions for discussion

- When are the roots of  $f(x) = 0$  real? When are the roots of  $f(x) = 0$  complex?
- For what values of  $b$  and  $c$  are the roots complex with real part 2?
- For which functions  $f(x)$  will the roots of  $f(x) = 0$  be  $2 \pm 3i$ ?
- How would you find a function  $f(x)$  with roots  $p \pm qi$  (for some  $p$  and  $q$ )?

**Problem** (Try the problem with pen and paper first then check it on your software)

The function  $f(z) = z^3 - 10z^2 + 34z - 40$  has a root  $z = 3 + i$ . Find the other two roots.

### Further Tasks

- Explain why the roots of a cubic with real coefficients will always form an isosceles triangle in the Argand diagram.
- Find a cubic  $f(z)$  where the roots of  $f(z)$  form an equilateral triangle in the Argand diagram.

# MEI GeoGebra Tasks for Further Maths: Pure

## Task 2 – Complex Numbers: Addition and subtraction in the Argand diagram

1. Use the complex number tool



The complex number tool can be found in the Points menu (*More on the Graphing app*).

to add  $z_1$  and  $z_2$  to the graphics view.

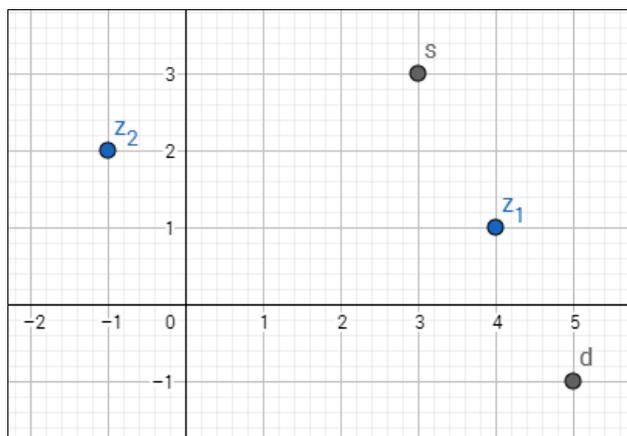
2. Add the sum and difference of  $z_1$  and  $z_2$  by typing in the input bar:

$$s = z_1 + z_2$$

$$d = z_1 - z_2$$

In Classic use underscore for  $z_1$ :  $z\_1$

In the graphing app use  $a_{\cdot}$  from the f(x) keyboard



### Questions

- What are the geometrical relationships between the original complex number and the sum and difference as represented in the Argand diagram?
- How do the sum and difference move when  $z_1$  and  $z_2$  are moved?

**Problem** (*Try the problem with pen and paper first then check it on your software*)

$z_1 = 3 + 2i$  and  $z_2 = -1 - i$ . Find  $z_1 + z_2$  and  $z_1 - z_2$ .

### Further Tasks

The conjugate can be found using: `conjugate(z_1)`

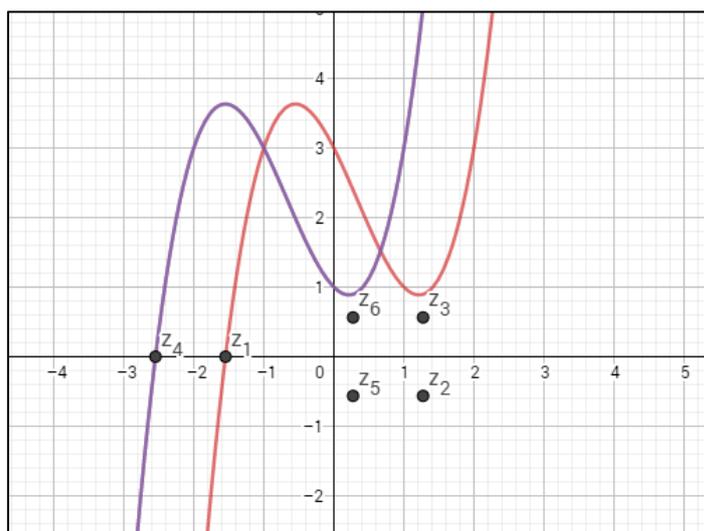
- For a complex number,  $z$ , and its conjugate,  $z^*$ , investigate the geometrical properties of  $z + z^*$  and  $z - z^*$ .
- Investigate the product and quotient of two complex numbers:  $z_1 z_2$  and  $\frac{z_1}{z_2}$ .

# MEI GeoGebra Tasks for Further Maths: Pure

## Task 3 – Roots of polynomial equations

1. In the input bar enter the function:  $f(x)=x^3-x^2+2x+3$
2. In the input bar enter: **ComplexRoot(f)**
3. In the input bar enter the function:  $g(x)=f(x+a)$   
If prompted click *Create Sliders*.
4. In the input bar enter: **Expand(g)**
5. In the input bar enter: **ComplexRoot(g)**

It is important that this is created as a function.



### Questions for discussion

- What is the relationship between the roots of  $f(x) = 0$  and the roots of  $f(x + a) = 0$ ?
- How could you find the expanded form of  $f(x + a)$ ?

**Problem** (Try the problem with pen and paper first then check it on your software)

The cubic equation  $x^3 - 2x^2 + 3x + 6 = 0$  has roots  $\alpha$ ,  $\beta$  and  $\gamma$ . Find a cubic equation with integer coefficients that has roots  $\alpha + 3$ ,  $\beta + 3$  and  $\gamma + 3$ .

### Further Tasks

- For a given cubic equation, with roots  $\alpha$ ,  $\beta$  and  $\gamma$ , find a cubic equation with integer coefficients that has roots  $k\alpha$ ,  $k\beta$  and  $k\gamma$  for different values of  $k$ .
- Investigate the coefficients of related quartic equations.

# MEI GeoGebra Tasks for Further Maths: Pure

## Task 4 – Matrices: Transformation matrices

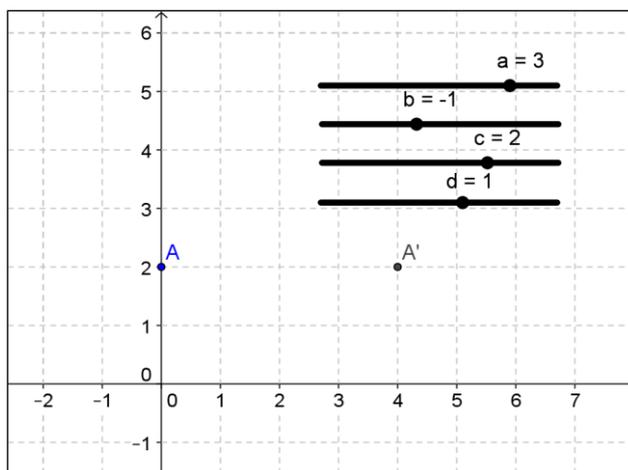
1. Create the matrix  $\mathbf{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$  by entering:  $\mathbf{M}=\{\{a,c\},\{b,d\}\}$  in the input bar.  
If prompted click *Create Sliders*.

2. Use the **Point** tool to add a point, **A**.



In Classic type  $\mathbf{A}'=\mathbf{M}\cdot\mathbf{A}$  or  $\mathbf{A}'=\mathbf{M} \mathbf{A}$

3. Create the image of A under the transformation M by entering  $\mathbf{A}'=\mathbf{M}\times\mathbf{A}$  in the input bar.



### Questions for discussion

- What is the relationship between  $A'$  and  $A$  when  $A$  is on the x-axis?
- What is the relationship between  $A'$  and  $A$  when  $A$  is on the y-axis?
- How can you use these relationships to find the position of  $A'$  for any point  $A$ ?

**Problem** (Try the problem with pen and paper first then check it on your software)

The image of the quadrilateral  $OABC$  under the transformation  $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$  is a square.  
 $O = (0,0)$ ,  $A = (2, -4)$  and  $B = (4,0)$ . Find the coordinates of  $C$ .

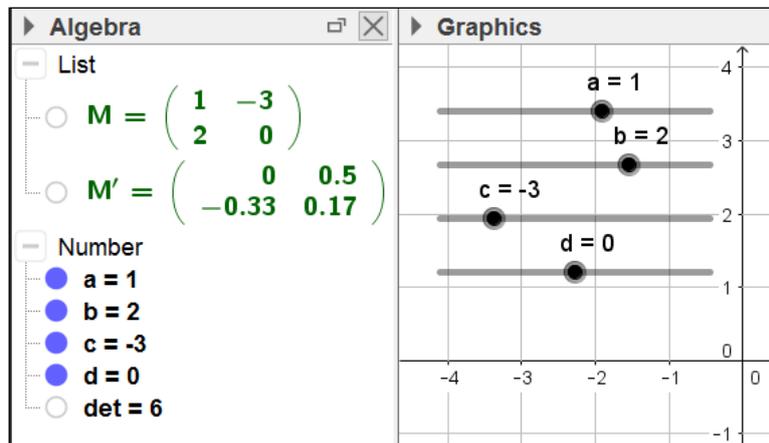
### Further Tasks

- Under the matrix  $\mathbf{M} = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$  the image of the point  $A = (-1,3)$  is  $A' = (2,4)$ .  
Find other matrices  $\mathbf{M}$  and points  $A$  such that  $A' = (2,4)$ .
- $A = (2,1)$  is an invariant point under the matrix  $\mathbf{M} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}$  because the image of the point  $A = (2,1)$  is  $A' = (2,1)$ . Find the invariant points of other matrices  $\mathbf{M}$ .

# MEI GeoGebra Tasks for Further Maths: Pure

## Task 5 – Matrices: Determinants and inverse matrices

1. Create the matrix  $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$  by entering: **M={{a,c},{b,d}}** in the input bar. If prompted click *Create Sliders*.
2. Find the determinant of **M** by entering: **det=Determinant(M)** in the input bar.
3. Find the inverse of **M** by entering: **M'=Invert(M)** in the input bar.



### Questions for discussion

- What is the relationship between the matrix, the determinant and the inverse?
- What is the answer when a matrix is multiplied by its inverse?
- Are there any matrices that don't have an inverse?

**Problem** (Try the question with pen and paper first then check it on your software)

For the matrices  $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$  and  $B = \begin{pmatrix} 2 & -1 \\ -5 & 4 \end{pmatrix}$  find  $A^{-1}$ ,  $B^{-1}$  and  $(AB)^{-1}$ .

### Further Tasks

- Investigate the determinants and inverse of matrices for standard transformations: reflections, rotations and stretches.

You might find it useful to draw a shape (**poly1**) with the polygon tool  and then use the command: **ApplyMatrix[M,poly1]**

- For other 2x2 matrices, A and B, investigate the relationship between  $A^{-1}$ ,  $B^{-1}$  and  $(AB)^{-1}$ .

# MEI GeoGebra Tasks for Further Maths: Pure

## Task 6 - Matrices: 3x3 matrices and transformations

GeoGebra Classic 3D view or GeoGebra 3D Graphing Calculator

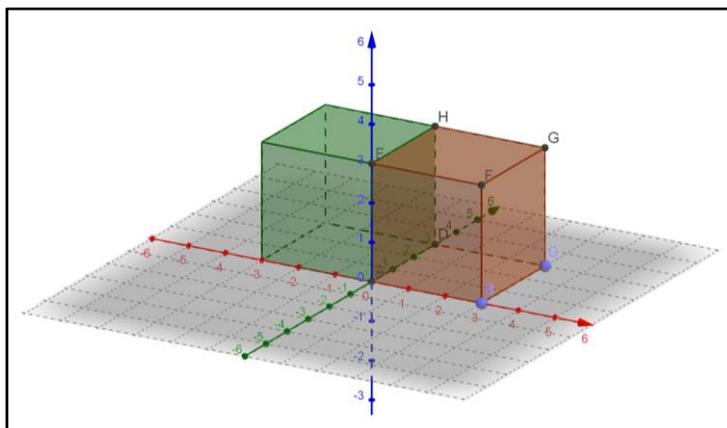
1. Create the matrix  $\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  by entering:  $\mathbf{M}=\{-1,0,0\},\{0,1,0\},\{0,0,1\}$  in the input bar.

2. Use the **Cube** tool and click on the origin followed by the point (3,0,0) to create a cube of side length 3.



The cube tool can be found in the 9<sup>th</sup> menu in Classic and the Solids tools on the app.

3. Create a list of the face names in the input bar by typing  $f= \{\text{faceABCD},\text{faceABFE},\text{faceADHE},\text{faceBCGF},\text{faceCDHG},\text{faceEFGH}\}$
4. Create the image of  $f$  under the transformation  $\mathbf{M}$  by typing: **ApplyMatrix** $[\mathbf{M},f]$  in the input bar.



### Questions for discussion

- What transformation does  $\mathbf{M}$  represent?
- What happens if you change the values on the leading diagonal of  $\mathbf{M}$ ?
- What transformation is represented by the Matrix  $\mathbf{M} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ?

**Problem** (Try the question with pen and paper first then check it on your software)

What transformation is represented by the matrix:  $\mathbf{M} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ ?

### Further Tasks

Construct some other matrices for standard transformations (reflections, rotations, enlargements) and check these by applying them to a cube.

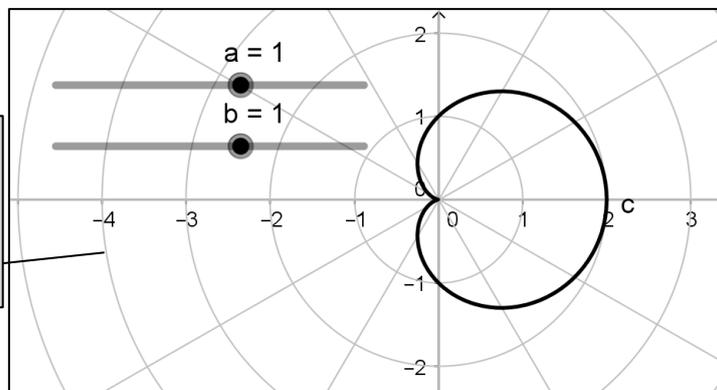
# MEI GeoGebra Tasks for Further Maths: Pure

## Task 7 – Polar curves

Enter the polar curve:  $r=a+\cos(b\theta)$   
If prompted select *Create Sliders*.

$\theta$  can be found in the  $\alpha$  menu in the input bar or using alt-T.

A polar grid can be displayed using the grid settings:



### Questions for discussion

- What is the maximum/minimum distance from the pole and for what values of  $\theta$  does this occur?
- How is this polar curve related to the Cartesian curve  $y = a + \cos bx$ ?

**Problem** (*Sketch the curves with pen and paper first then plot them on your software*)

Sketch the following curves:

$$r = 2 + \cos \theta$$

$$r = 2$$

$$r = 1 + \sin 2\theta$$

$$r = 3 + \cos \frac{\theta}{2}$$

$$r = 3 - 2 \sin \theta$$

### Further Tasks

- For what values of  $\theta$  does  $r$  take its maximum and minimum values? How can these be deduced from the polar equation?
- For which parts of the graph does  $r$  take negative values? What are the conditions such that  $r = a + b \cos \theta$  and  $r = a + b \sin \theta$  don't take negative values?
- The default setting is to plot values of  $\theta$  from 0 to  $2\pi$  (this can be changed by selecting the function). Are there any curves for which this results in the same graph being traced over again? Are there any graphs for which the graph is incomplete using this range?

# MEI GeoGebra Tasks for Further Maths: Pure

## Task 8 – Vectors: Intersection of three planes

*GeoGebra Classic 3D view or GeoGebra 3D Graphing Calculator*

1. In the input bar enter the equations of three planes:

$$x - 2y + 4z = 4$$

$$x + y - z = 2$$

$$x + 3y + z = 6$$

7<sup>th</sup> menu in Classic or  
Basic Tools on the app.

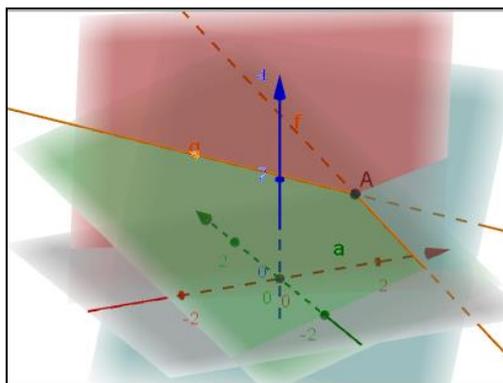
2. Use the **Intersect two surfaces** tool to find the lines of intersection of two of the pairs of planes.



3. Use the **Intersect** tool to find the point of intersection of these lines.



2<sup>nd</sup> menu in Classic or  
Points Tools on the app.



### Question for discussion

- How can you use a matrix to find the point of intersection?

**Problem** (Try just using the inverse matrix function on your calculator then check it on your software)

Find the point of intersection of the planes:

$$5x - y + 2z = 5$$

$$x - 5y - 2z = -11$$

$$x + y + 2z = 3$$

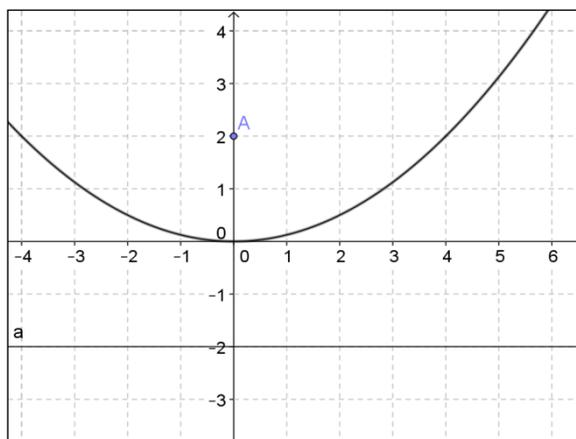
### Further Tasks

- Investigate how many ways three planes can be orientated so that they do not have a unique point of intersection.
- For each of these cases determine whether there are no points of intersection or infinitely many.

# MEI GeoGebra Tasks for Further Maths: Pure

## Task 9 – Parabolas, hyperbolas and ellipses

1. Use **New Point** (2<sup>nd</sup> menu)  to add a point on the y-axis, **A**.
2. In the input bar enter:  $y=-y(A)$
3. Use **Parabola** (7<sup>th</sup> menu)  to create a parabola with focus **A** and directrix  $y=-y(A)$ .



### Question

What is the relationship between the equation of the curve and position of **A**?

**Problem** (Try the problem with pen and paper first then check it on your software)

Show that the line joining the points **A**: (4,4) and **B**:  $(-1, \frac{1}{4})$  passes through the focus of  $x^2 = 4y$ .

Show that the tangents to  $x^2 = 4y$  at the points **A**: (4,4) and **B**:  $(-1, \frac{1}{4})$  meet at right-angles on the directrix to the parabola.

### Further Tasks

On the 7<sup>th</sup> menu there are tools for creating an Ellipse:  and a Hyperbola: 

These are based the position of the two foci (**A** and **B**) and a point on the curve **C**.

Find positions for **A**, **B** and **C** for the following curves:

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

$$\frac{x^2}{25} + \frac{y^2}{34} = 1$$

$$y^2 - \frac{x^2}{8} = 1$$

$$\frac{x^2}{8} - \frac{y^2}{8} = 1$$

$$y = \frac{1}{x}$$

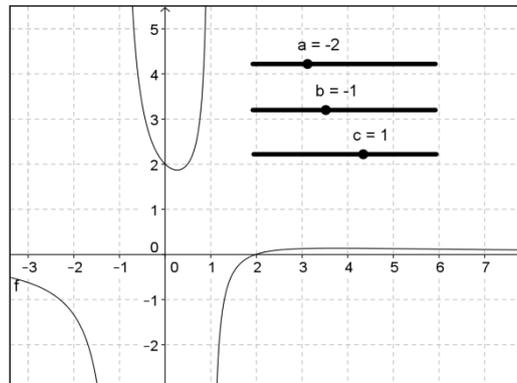
$$y = \frac{k}{x}$$

# MEI GeoGebra Tasks for Further Maths: Pure

## Task 10 – Sketching the curves of rational Functions

In the input bar enter the function:  $f(x)=(x+a)/((x+b)(x+c))$ .

If prompted select *Create Sliders*.

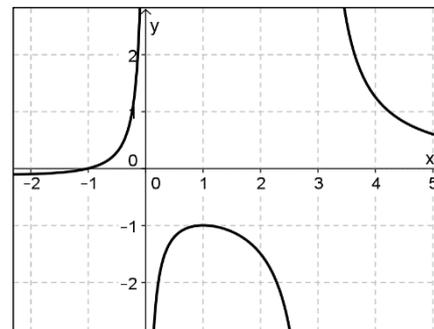
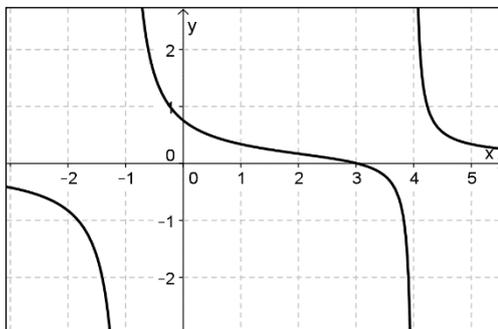


### Questions

- What point on the curve does the value of  $a$  give you?
- What is the relationship between the shape of the curve and the values of  $b$  and  $c$ ?

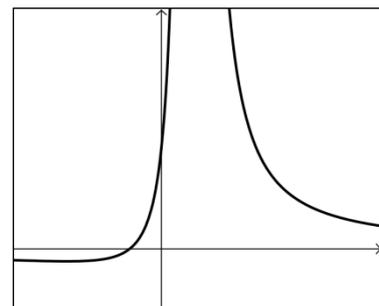
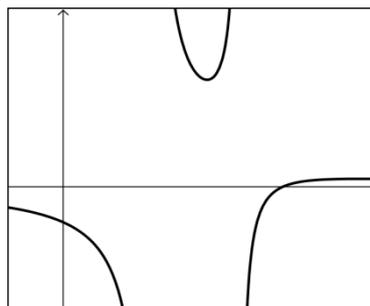
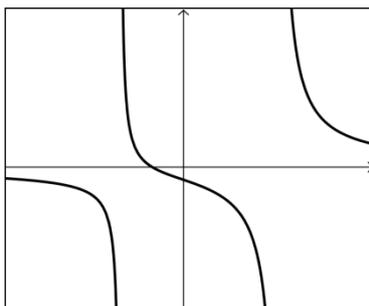
**Problem** (Try the question with pen and paper first then check it on your software)

Find the values of  $a$ ,  $b$  and  $c$  for the following curves:



### Further Task

Find conditions on  $a$ ,  $b$  and  $c$  so that the curve will have one of these general shapes.



# MEI GeoGebra Tasks for Further Maths: Pure

## Teacher guidance

### Using these tasks

These tasks are designed to help students in understanding mathematical relationships better through exploring dynamic constructions. They can be accessed using the computer-based version of GeoGebra or the tablet/smartphone app. Each task instruction sheet is reproducible on a single piece of paper and they are designed to be an activity for a single lesson or a single homework task (approximately).

Most of these tasks have been designed with the following structure –

- **Construction:** step-by-step guidance of how to construct the objects in GeoGebra. Students will benefit from learning the rigorous steps need to construct objects and this also removes the need to make prepared files available to them. If students become confident with using GeoGebra they can be encouraged to add additional objects to the construction to aid their exploration.
- **Questions for discussion:** This discussion can either be led as a whole class activity or take place in pairs/small groups. The emphasis is on students being able to observe mathematical relationships by changing objects on their screen. They should try to describe what happens, and explain why.
- **Problem:** Students are expected to try the problem with pen and paper first then check it on their software. The purpose is for them to formalise what they have learnt through exploration and discussion and apply this to a “standard” style question. Students could write-up their answers to the discussion questions and their solution to this problem in their notes to help consolidate their learning and provide evidence of what they’ve achieved. This problem can be supplemented with additional textbook questions at this stage if appropriate.
- **Further Tasks:** Extension activities with less structure for students who have successfully completed the first three sections.

### Task 1 – Complex Numbers: Roots of Quadratic Equations

In this task students investigate the link between the coefficients of a quadratic equation and its roots, considering when they are either real or complex. Students should be encouraged to use the completing the square method for solving quadratic equations.

*Problem solution*

$$z = 4, z = 3 + i, z = 3 - i.$$

### Task 2 – Complex Numbers: Addition and subtraction in the Argand Diagram

In this task students investigate the geometry of the sum and product of complex numbers and their conjugates. This can then be extended to products and quotients if appropriate.

*Problem solution*

$$z_1 + z_2 = 2 + 3i \text{ and } z_1 - z_2 = 4 + i .$$

# MEI GeoGebra Tasks for Further Maths: Pure

## Task 3 – Roots of polynomial equations

In this task students investigate the link between the roots and coefficients of related polynomials. They should be encouraged to try finding the expanded form of  $f(x + a)$  by both expanding the polynomial and considering the sums and products of the roots. Students who have difficulty with a cubic could start with a quadratic.

*Problem solution*

$$x^3 - 11x^2 + 42x - 48 = 0$$

## Task 4 – Matrices: Transformation matrices

By investigating the transformation of points on the axes, such as (1,0) and (0,1), students can observe the relationship between the columns of the matrix and the transformation. This can be used to find the x and y coordinates of the image of any point under the transformation.

*Problem solution*

$$C = (2,4)$$

## Task 5 – Matrices: Determinants and inverse matrices

This task can be used to reinforce the link between the determinant and the inverse matrix. Students should be encouraged to find some singular matrices and investigate transformations under these matrices.

*Problem solution*

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{pmatrix}, \mathbf{B}^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{1}{3} \\ \frac{5}{3} & \frac{2}{3} \end{pmatrix}, \text{ and } (\mathbf{AB})^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} \\ \frac{3}{5} & -\frac{2}{15} \end{pmatrix}. \text{ Note that } (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

## Task 6 - Matrices: 3x3 Matrices and Transformations

In this task students should be encouraged to keep their matrices simple initially. Cases where one of the dimensions remains unchanged are useful to help build understanding.

Matrices are entered into GeoGebra using a separate list for each row with the list of rows being interpreted as a matrix.

For example, the matrix  $\mathbf{M} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$  can be created by entering:

$\mathbf{M}=\{\{1,4,7\},\{2,5,8\},\{3,6,9\}\}$  in the input bar.

*Problem solution*

Reflection in the plane  $x = -z$ .

# MEI GeoGebra Tasks for Further Maths: Pure

## Task 7 – Polar curves

To display a polar grid:

- GeoGebra Classic 5: right-click in the graphics area and select **Graphics** (the last option), on the grid option select **Grid type: Polar**.
- GeoGebra Classic 6: the polar grid can be found in the graphics style bar under the grid settings.
- GeoGebra Graphing Calculator app: the polar grid can be found in the settings under **Show Grid**.

## Task 8 – Vectors: Intersection of three planes

Students should be encouraged to show how the inverse to the matrix  $\mathbf{M} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & -1 \\ 1 & 3 & 1 \end{pmatrix}$  can be used to find the point of intersection.

*Problem solution*

$$A = (2, 3, -1)$$

## Task 9 – Parabolas, Hyperbolas and Ellipses

Students should be encouraged to make the link between the  $x^2 = 4ay$  form of the parabola and the explicit equations of the curve that they are likely to be familiar with:  $y = \frac{x^2}{4a}$ .

Students could try to generalise the result in the problem.

## Task 10 – Curve sketching: Rational Functions

In this task students investigate the effect of the parameters on graphs of the rational functions:  $y = \frac{x+a}{(x+b)(x+c)}$

*Problem solutions*

$$y = \frac{x-3}{(x-4)(x+1)}, \quad y = \frac{x+1}{x(x-3)}$$