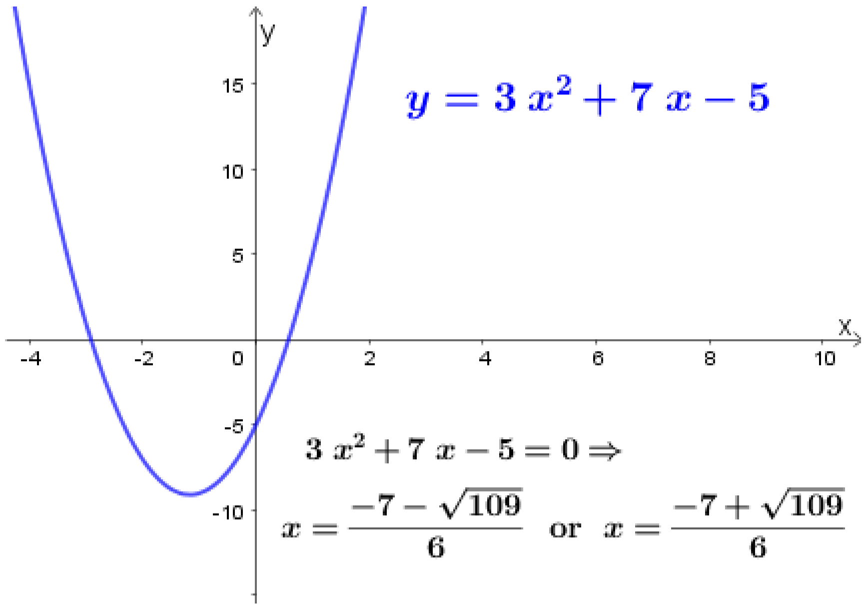
MEI Maths Item of the Month

February 2021
It's odd but is it rational?

The graph with equation $y = 3x^2 + 7x - 5$ and the roots of the equation $3x^2 + 7x - 5 = 0$ are shown in the image below.



If the values of a, b and c are all odd integers will the equation $ax^2 + bx + c = 0$ ever have rational roots?

If the values of a, b and c are integers there are 8 possible combinations for these to be odd or even. For which of these combinations is it possible for the equation $ax^2 + bx + c = 0$ to have rational roots?

Solution

If the values of a, b and c are all odd integers will the equation $ax^2 + bx + c = 0$ ever have rational roots?

No!

To prove this, assume there is a rational number $\frac{p}{q}$ (already reduced to its lowest terms) which satisfies the equation.

Since $\frac{p}{q}$ is in its lowest terms then at least one of p and q must be odd.

Therefore $a\left(\frac{p^2}{q^2}\right) + b\left(\frac{p}{q}\right) + c = 0$ and multiplying throughout by q^2 gives

$ap^2 + bpq + cq^2 = 0$

- If both p and q are odd then the equation says the sum of three odd numbers is 0 and this is impossible.
- If exactly one of p and q is odd then the equation says the sum of one odd and two even numbers is 0 and this is impossible.

Therefore if a, b and c are odd, any real roots of $ax^2+bx+c=0$ are irrational.

If the values of a, b and c are integers there are 8 possible combinations for these to be odd or even. For which of these combinations is it possible for the equation $ax^2+bx+c=0$ to have rational roots?

All other combinations are possible.

a	b	c	Example
Odd	Odd	Odd	Impossible!
Odd	Odd	Even	$(x-1)(x-2)=0$
Odd	Even	Odd	$(x-1)(x+1)=0$
Odd	Even	Even	$(x-2)(x+2)=0$
Even	Odd	Odd	$(2x-1)(x+1)=0$
Even	Odd	Even	$(2x-1)(x+2)=0$
Even	Even	Odd	$(2x-1)(2x+1)=0$
Even	Even	Even	$2x(x+1)=0$

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