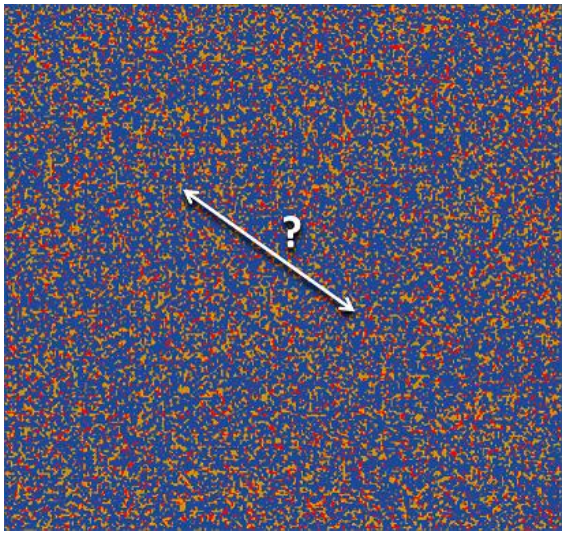
MEI Maths Item of the Month

March 2021
Colouring the plane with three colours

Every point in the plane is coloured either red, blue or yellow.



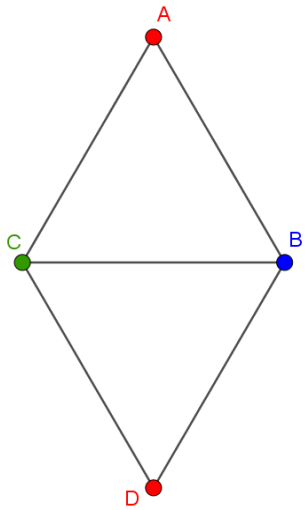
Prove that there must be two points of the same colour that are exactly one unit apart.

Solution

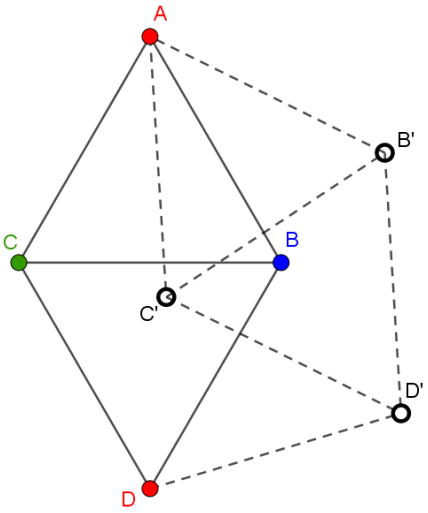
The result can be proved using a proof by contradiction.

Assume there is a colouring of the plane where no two points distance one unit apart are the same colour. In this colouring, take three points, A, B and C, at the corners of an equilateral triangle of side length one unit. Then these must all be coloured differently. Say A is red.

Reflect A in the line BC to create a rhombus ABDC as shown. D can't be the same colour as B or C since it is one unit from each of these points, so D is red.



Now rotate the rhombus ABDC anticlockwise about A until D', the image of D, is in a position that is one unit from D. What colour is D'?



Since B' and C', both one unit from A, must be coloured differently from A and from each other (since they are one unit apart) then one of these must be blue and the other green. Now D' is one unit from each of D (red), B' and C' (blue and green) and so it can't be any of those colours. We have a contradiction. Therefore, in any colouring there must be points one unit apart that are the same colour.