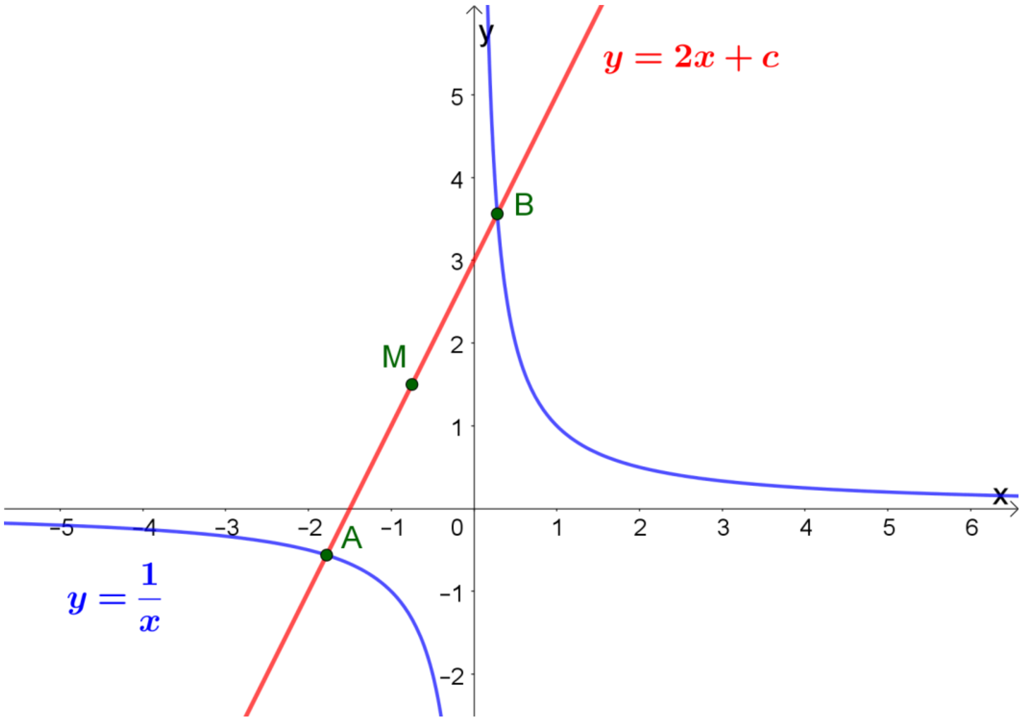
MEI Maths Item of the Month

September 2020
Path of the midpoint



M is the midpoint of the points of intersection of $y = \frac{1}{x}$ and $y = 2x + c$.

What path does M trace as c varies?

Solution

M traces the path $y = -2x$.

The points of intersection A and B have x-coordinates given by $\frac{1}{x} = 2x + c$.
$\frac{1}{x} = 2x + c$ can be rearranged to $2x^2 + cx - 1 = 0$ $(x \neq 0)$.

The roots of $2x^2 + cx - 1 = 0$ are at $x = \frac{-c \pm \sqrt{c^2+8}}{4}$.
As the points A and B lie on $y = \frac{1}{x}$ they have y-coordinates $y = \frac{4}{-c \pm \sqrt{c^2+8}}$.

The x-coordinate of the midpoint M is: $x = -\frac{c}{4}$.

The y-coordinate of the midpoint M is: $y = \frac{1}{2}\left(\frac{4}{-c+\sqrt{c^2+8}} + \frac{4}{-c-\sqrt{c^2+8}}\right)$
$\Rightarrow y = 2\left(\frac{-c+\sqrt{c^2+8}+(-c-\sqrt{c^2+8})}{(-c+\sqrt{c^2+8})(-c-\sqrt{c^2+8})}\right) \Rightarrow y = 2\left(\frac{-2c}{-8}\right) \Rightarrow y = \frac{c}{2}$.

Therefore the midpoint M is at the point $\left(\frac{-c}{4}, \frac{c}{2}\right)$ which is the set of points for which $y = -2x$.