OCR
Oxford Cambridge and RSA

Wednesday 24 June 2015 – Morning

A2 GCE MATHEMATICS (MEI)

4777/01 Numerical Computation

Candidates answer on the Answer Booklet.

OCR supplied materials:
- 12 page Answer Booklet (OCR12) (sent with general stationery)
- MEI Examination Formulae and Tables (MF2)
- Graph paper

Other materials required:
- Scientific or graphical calculator
- Computer with appropriate software and printing facilities

Duration: 2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the Answer Booklet. Please write clearly and in capital letters.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- Additional sheets, including computer print-outs, should be fastened securely to the Answer Booklet.
- Do **not** write in the bar codes.

COMPUTING RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities throughout the examination.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.
  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the formulae in the cells as well as the values in the cells.
  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 **(i)** Suppose that the equation $x = \mathrm{g}(x)$ has a root α.

State the condition for the iteration $x_{r+1} = \mathrm{g}(x_r)$ to converge to α. (You may assume that a suitable starting value is used.)

Obtain the relaxed iteration

$$x_{r+1} = (1-\lambda)x_r + \lambda \mathrm{g}(x_r).$$

Find, in terms of α, the best value of λ. How would a value of λ be chosen in practice? **[5]**

(ii) Show that the equation $2\tan x = \frac{1}{x-1.5}$ has a root in the interval $[1.5, \frac{\pi}{2}]$.

Show numerically that the iteration based on the rearrangement $x = \arctan\left(\frac{0.5}{x-1.5}\right)$ diverges from this root. **[6]**

(iii) Show that the corresponding relaxed iteration with $\lambda = 0.5$ converges. Obtain the root correct to 5 decimal places. **[4]**

(iv) Determine numerically the value of λ, correct to 2 decimal places, that gives fastest convergence. **[3]**

(v) Solve the equation $k\tan x = \frac{1}{x-1.5}$ for $k = 0.5, 1, 1.5, 2, 2.5, 3$, and $1.5 < x < \frac{\pi}{2}$. Provide a sketch of the root as a function of k. **[6]**

2 **(i)** The trapezium rule, using n intervals of equal width h, is used to find an estimate T_n of the integral

$$I = \int_a^b \mathrm{f}(x)\,\mathrm{d}x.$$

You are given that the global error in T_n is of the form

$$A_2h^2 + A_4h^4 + A_6h^6 + \ldots,$$

where the coefficients A_2, A_4, A_6, ... are independent of n and h.

Show that $T_n{}^* = \frac{1}{3}(4T_{2n} - T_n)$ is an estimate of I with global error of order h^4.

Write down, without proof, an expression, $T_n{}^{**}$, in terms of $T_{2n}{}^*$ and $T_n{}^*$, that represents an estimate of I with global error of order h^6. **[6]**

Romberg's method is to be used to find values of the integral J, where

$$J = \int_a^2 \sqrt{\mathrm{e}^x - \mathrm{e}^{-x}}\,\mathrm{d}x,$$

for various values of a.

(ii) For the case $a = 1.5$, find the value of J correct to 6 decimal places. Show that the T_n, $T_n{}^*$ and $T_n{}^{**}$ estimates have the global errors indicated by the theory in part **(i)**. **[12]**

(iii) For the case $a = 0$, show that the global errors are not as theory would indicate. By considering the gradient of the curve $y = \sqrt{\mathrm{e}^x - \mathrm{e}^{-x}}$ when $x = 0$, explain why this is the case. **[6]**

3 The second order differential equation

$$y'' + xyy' = 1 + y^2, \text{ where } y = 1 \text{ and } y' = 0.5 \text{ when } x = 1,$$

is to be solved numerically using finite difference methods.

(i) Using central difference approximations, show that, in the usual notation,

$$y_{r+1}(2 + hx_r y_r) = 4y_r + 2h^2\left(1 + y_r^2\right) - y_{r-1}(2 - hx_r y_r),$$

and that

$$y_1 = \tfrac{1}{4}(4 + 2h + 3h^2). \qquad (*)$$

[6]

(ii) Obtain values of y when $x = 10$ for $h = 0.1, 0.05, 0.025, 0.0125$. Show that the method of solution is second order. **[10]**

(iii) Suppose now that (*) is modified to

$$y_1 = \tfrac{1}{4}(4 + 2h)$$

on the grounds that, for small h, the term in h^2 will be negligible. Investigate the order of the method after this change. **[8]**

4 The system of equations with augmented matrix

$$\left(\begin{array}{cccc|c} 5 & 3 & -2 & 7 & a \\ 11 & -6 & -5 & 2 & b \\ -4 & 3 & 3 & 4 & c \\ 6 & 2 & -4 & 5 & d \end{array}\right)$$

is to be solved using Gaussian elimination with partial pivoting.

(i) Set up a spreadsheet to find the solution in the case $a = 1$, $b = c = d = 2$. Verify that your solution is correct.

Explain, with reference to your solution, what is meant by partial pivoting. Explain also why it is used. **[16]**

(ii) From your working in part **(i)** obtain the determinant of the coefficient matrix. Explain how the sign is calculated. **[3]**

(iii) By choosing suitable values of a, b, c, d, use the spreadsheet developed in part **(i)** to find the inverse of the coefficient matrix. **[5]**

END OF QUESTION PAPER