OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4758

Differential Equations

Thursday **15 JUNE 2006** Afternoon 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- There is an **insert** for use in Question **3**.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\ \mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 3 printed pages, 1 blank page and an insert.

1 The displacement x at time t of an oscillating system from a fixed point is given by

$$\ddot{x} + 2\lambda\dot{x} + 5x = 0,$$

where $\lambda \geqslant 0$.

(i) For what value of λ is the motion simple harmonic? State the general solution in this case. [3]

(ii) Find the range of values of λ for which the system is under-damped. [3]

Consider the case $\lambda = 1$.

(iii) Find the general solution of the differential equation. [3]

When $t = 0$, $x = x_0$ and $\dot{x} = 0$, where x_0 is a positive constant.

(iv) Find the particular solution. [4]

(v) Find the least positive value of t for which $x = 0$. [3]

Now consider the case $\lambda = 3$ with the same initial conditions.

(vi) Find the particular solution and show that it is never zero for $t > 0$. [8]

2 The positive quantities x, y and z are related and vary with time t, where $t \geqslant 0$. The value of x is described by the differential equation

$$\frac{dx}{dt} + 2x = t + 1.$$

When $t = 0, x = 1$.

(i) Solve the equation to find x in terms of t. [9]

The quantity y is related to x by the differential equation $2x\frac{dy}{dx} = y$. When $t = 0, y = 4$.

(ii) Solve the equation to find y in terms of x. Hence express y in terms of t. [5]

The quantity z is related to x by the differential equation $x\frac{dz}{dx} + 2z = 6x$. When $t = 0, z = 3$.

(iii) Solve this equation for z in terms of x. Calculate the values of x, y and z when $t = 1$, giving your answers correct to 3 significant figures. [10]

3 **Answer parts (i) and (ii) on the insert provided.**

Two spherical bodies, Alpha and Beta, each of radius 1000 km, are in deep space. The point A is on the surface of Alpha, and the point B is on the surface of Beta. These points are the closest points on the two bodies and the distance AB has the constant value of 8000 km.

A probe is fired from A at a speed of V_0 km s^{-1} in an attempt to reach B, travelling in a straight line. At time t seconds after firing, the displacement of the probe from A is x km, and the velocity of the probe is v km s^{-1}.

The equation of motion for the probe is

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{(9000 - x)^2} - \frac{1}{(1000 + x)^2}.$$

This differential equation is to be investigated first by means of a tangent field, shown on the insert.

(i) Show that the direction indicators are parallel to the v-axis when $v = 0$ $(x \neq 4000)$. Show also that the direction indicators are parallel to the x-axis when $x = 4000$ $(v \neq 0)$. Hence complete the tangent field on the insert, excluding the point $(4000, 0)$. [6]

(ii) Sketch the solution curve through $(0, 0.025)$ and the solution curve through $(0, 0.05)$. Hence state what happens to the probe when the speed of projection is

(A) 0.025 km s^{-1},

(B) 0.05 km s^{-1}. [6]

(iii) Solve the differential equation to find v^2 in terms of x and V_0. [6]

(iv) Given that the probe reaches B, state the value of x at which v^2 is least. Hence find from your solution in part **(iii)** the range of values of V_0 for which the probe reaches B. [6]

4 The simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x - y + 3$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 5x - 4y + 18$$

are to be solved for $t \geqslant 0$.

(i) Show that $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\dfrac{\mathrm{d}x}{\mathrm{d}t} - 3x = -6$. [6]

(ii) Find the general solution for x in terms of t. Hence obtain the corresponding general solution for y. [9]

(iii) Given that $x = 4$, $y = 17$ when $t = 0$, find the particular solutions for x and y and sketch a graph of each solution. [9]

Candidate Name	Centre Number	Candidate Number

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Differential Equations

INSERT

Thursday **15 JUNE 2006** Afternoon 1 hours 30 minutes

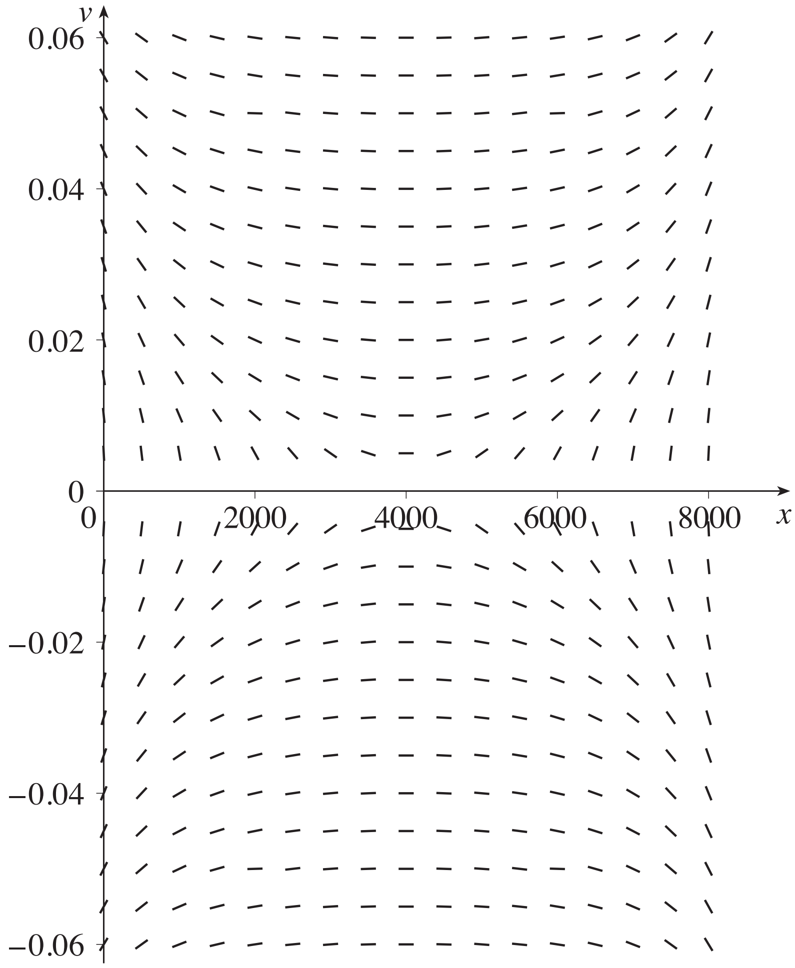
INSTRUCTIONS TO CANDIDATES

- This **insert** should be used in Question **3.**
- Write your name, centre number and candidate number in the spaces provided at the top of this page and attach it to your answer booklet.

This insert consists of 2 printed pages.

[Turn over

Insert for use with Question 3



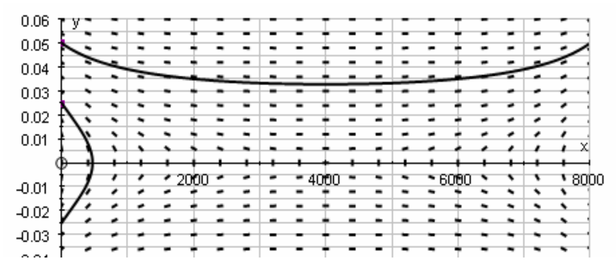
Mark Scheme 4758
June 2006

1(i)	$\lambda = 0$	B1		
	$x = A\cos\sqrt{5}t + B\sin\sqrt{5}t$	M1	$\cos\sqrt{5}t$ or $\sin\sqrt{5}t$ or $A\cos\omega t + B\sin\omega t$ seen or GS for their λ	
		A1		
				3
(ii)	$(2\lambda)^2 - 4\cdot 5 < 0$	M1	Use of discriminant	
		A1	Correct inequality	
	$0 < \lambda < \sqrt{5}$	A1	Accept lower limit omitted or $-\sqrt{5}$	
				3
(iii)	$\alpha^2 + 2\alpha + 5 = 0$	M1	Auxiliary equation	
	$\alpha = -1 \pm 2j$	A1		
	$x = e^{-t}\left(C\cos 2t + D\sin 2t\right)$	F1	CF for their roots	
				3
(iv)	$x_0 = C$	M1	Condition on *x*	
	$\dot{x} = -e^{-t}\left(C\cos 2t + D\sin 2t\right) + e^{-t}\left(-2C\sin 2t + 2D\cos 2t\right)$	M1	Differentiate (product rule)	
	$0 = -C + 2D$	M1	Condition on $\dot{x}$	
	$D = \frac{1}{2}x_0$			
	$x = x_0\,e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$	A1	cao	
				4
(v)	$\cos 2t + \frac{1}{2}\sin 2t = 0$	M1		
	$\tan 2t = -2$	M1		
	$t = 1.017$	A1	cao	
				3
(vi)	$\alpha^2 + 6\alpha + 5$	M1	Auxiliary equation	
	$\alpha = -1, -5$	A1		
	$x = E\,e^{-t} + F\,e^{-5t}$	F1	CF for their roots	
	$x_0 = E + F$	M1	Condition on *x*	
	$\dot{x} = -E\,e^{-t} - 5F\,e^{-5t}$			
	$0 = -E - 5F$	M1	Condition on $\dot{x}$	
	$E = \frac{5}{4}x_0, F = -\frac{1}{4}x_0$			
	$x = \frac{1}{4}x_0\left(5\,e^{-t} - e^{-5t}\right)$	A1	cao	
	$x = \frac{1}{4}x_0\,e^{-t}\left(5 - e^{-4t}\right)$	M1	Attempt complete method	
	$t > 0 \Rightarrow 5 > e^{-4t}, x_0 > 0, e^{-t} > 0 \Rightarrow x > 0$ i.e. never zero	E1	Fully justified (only $\neq 0$ required)	
				8

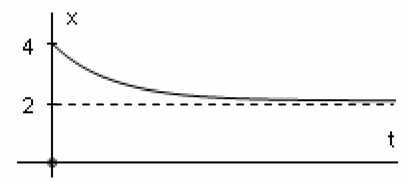
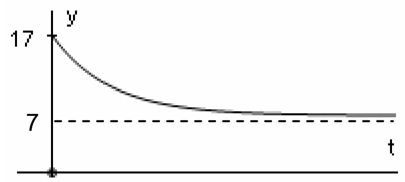
2(i)	$\lambda + 2 = 0 \Rightarrow \lambda = -2$	M1		
	CF $x = A\mathrm{e}^{-2t}$	A1		
	PI $x = at + b$	B1		
	$a + 2(at+b) = t + 1$	M1	Differentiate and substitute	
	$2a = 1,\ a + 2b = 1$	M1	Compare	
	$a = \frac{1}{2}, b = \frac{1}{4}$	A1		
	$x = \frac{1}{2}t + \frac{1}{4} + A\mathrm{e}^{-2t}$	F1	CF + PI	
	$t = 0, x = 1 \Rightarrow 1 = \frac{1}{4} + A$	M1	Condition on x	
	$x = \frac{1}{2}t + \frac{1}{4} + \frac{3}{4}\mathrm{e}^{-2t}$	F1	Follow a non-trivial GS	
	Alternatively:			
	$I = \exp\left(\int 2\ \mathrm{d}t\right) = \mathrm{e}^{2t}$	M1 A1	Integrating factor	
	$\mathrm{e}^{2t}\frac{\mathrm{d}x}{\mathrm{d}t} + 2\mathrm{e}^{2t}x = \mathrm{e}^{2t}(t+1)$	B1	Multiply DE by their I	
	$\mathrm{e}^{2t}x = \int \mathrm{e}^{2t}(t+1)\,\mathrm{d}t$	M1	Attempt integral	
	$= \frac{1}{2}\mathrm{e}^{2t}(t+1) - \int \frac{1}{2}\mathrm{e}^{2t}\,\mathrm{d}t$	M1	Integration by parts	
	$\mathrm{e}^{2t}x = \frac{1}{2}\mathrm{e}^{2t}(t+1) - \frac{1}{4}\mathrm{e}^{2t} + A$	A1		
	$x = \frac{1}{2}t + \frac{1}{4} + A\mathrm{e}^{-2t}$	F1	Divide by their I (must also divide constant)	
	$t = 0, x = 1 \Rightarrow 1 = \frac{1}{4} + A$	M1	Condition on x	
	$x = \frac{1}{2}t + \frac{1}{4} + \frac{3}{4}\mathrm{e}^{-2t}$	F1	Follow a non-trivial GS	
				9
(ii)	$\frac{2}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$	M1	Separate	
	$\int \frac{2}{y}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x$	M1	Integrate	
	$2\ln y = \ln x + c$			
	$y = B\sqrt{x}$	M1	Make y subject, dealing properly with constant	
	$(t=0), x = 1, y = 4 \Rightarrow y = 4\sqrt{x}$	M1	Condition	
	$y = 4\sqrt{\frac{1}{2}t + \frac{1}{4} + \frac{3}{4}\mathrm{e}^{-2t}}$	F1	$y = 4\sqrt{}$(their x in terms of t)	
				5
(iii)	$\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{2}{x}z = 6$	M1	Divide DE by x	
	$I = \exp\left(\int \frac{2}{x}\,\mathrm{d}x\right)$	M1	Attempt integrating factor	
	$= x^2$	A1	Simplified	
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 z\right) = 6x^2$	F1	Follow their integrating factor	
	$x^2 z = 2x^3 + C$	A1		
	$z = 2x + Cx^{-2}$	F1	Divide by their I (must also divide constant)	
	$(t=0), x = 1, z = 3 \Rightarrow C = 1$	M1	Condition on z	
	$z = 2x + x^{-2}$	A1	cao (in terms of x)	
	$t = 1 \Rightarrow x = 0.852$			
	$y = 3.69$	B1	Any 2 values (at least 3sf)	
	$z = 3.08$	B1	All 3 correct (and 3sf)	
				10

3(i)	$\frac{dv}{dx} = \frac{1}{v} f(x)$ so $\left(\text{unless } f(x)=0\right), v \to 0 \Rightarrow \frac{dv}{dx} \to \pm\infty$	M1	Consider $\frac{dv}{dx}$ or $\frac{dx}{dv}$ when v = 0, but not if $\frac{dv}{dx} = 0$	
	i.e. gradient parallel to v-axis (vertical)	E1	Must conclude about direction	
	$x = 4000 \Rightarrow v\frac{dv}{dx} = \frac{1}{5000^2} - \frac{1}{5000^2} = 0$	M1	Consider $\frac{dv}{dx}$ when x = 4000	
	so if $v \neq 0$ then gradient parallel to x-axis (horizontal)	E1	Must conclude about direction	
		M1	Add to tangent field	
		A1	Several vertical direction indicators on x-axis	6

(ii)



(ii)	[graph above]	M1	Attempt one curve	
		A1		
		M1	Attempt second curve	
		A1		
	$V_0 = 0.05 \Rightarrow$ probe reaches B	B1	Must be consistent with their curve	
	$V_0 = 0.025 \Rightarrow$ probe returns to A	B1	Must be consistent with their curve N.B. Cannot score these if curve not drawn	6
(iii)	$\int v \, dv = \int \left((9000-x)^{-2} - (1000+x)^{-2}\right) dx$	M1	Separate	
	$\frac{1}{2}v^2 = \frac{1}{9000-x} + \frac{1}{1000+x} + c$	M1	Integrate	
		B1	LHS	
		A1	RHS	
	$\frac{1}{2}V_0^2 = \frac{1}{9000} + \frac{1}{1000} + c$	M1	Condition	
	$v^2 = \frac{2}{9000-x} + \frac{2}{1000+x} + V_0^2 - \frac{1}{450}$	A1		6
(iv)	minimum when x = 4000	B1	Clearly stated	
	$v_{min}^2 = \frac{2}{5000} + \frac{2}{5000} + V_0^2 - \frac{1}{450}$	M1	Substitute their x into v or v^2	
		F1	Their v^2 or v when x = 4000	
	need $v_{min}^2 > 0$	M1	For $v_{min}^2 > 0$	
	$v_{min}^2 > 0$ if $V_0^2 > \frac{1}{450} - \frac{4}{5000}$	M1	Attempt inequality for V_0^2	
	$V_0 > 0.0377$	A1	cao	6

4(i)	$\ddot{x} = 2\dot{x} - \dot{y}$	M1	Differentiate first equation	
	$= 2\dot{x} - (5x - 4y + 18)$	M1	Substitute for $\dot{y}$	
	$y = 2x + 3 - \dot{x}$	M1	y in terms of $x, \dot{x}$	
	$\ddot{x} = 2\dot{x} - 5x + 4(2x + 3 - \dot{x}) - 18$	M1	Substitute for y	
	$\ddot{x} + 2\dot{x} - 3x = -6$	E1	LHS	
		E1	RHS	
				6
(ii)	$\lambda^2 + 2\lambda - 3 = 0$	M1	Auxiliary equation	
	$\lambda = 1$ or -3	A1		
	CF $x = A\mathrm{e}^{-3t} + B\mathrm{e}^{t}$	F1	CF for their roots	
	PI $x = a$	B1	Constant PI	
	$-3a = -6 \Rightarrow a = 2$	B1	PI correct	
	$x = 2 + A\mathrm{e}^{-3t} + B\mathrm{e}^{t}$	F1	Their CF + PI	
	$y = 2x + 3 - \dot{x}$	M1	y in terms of $x, \dot{x}$	
	$= 4 + 2A\mathrm{e}^{-3t} + 2B\mathrm{e}^{t} + 3 - (-3A\mathrm{e}^{-3t} + B\mathrm{e}^{t})$	M1	Differentiate x and substitute	
	$y = 7 + 5A\mathrm{e}^{-3t} + B\mathrm{e}^{t}$	A1	Constants must correspond with those in x	
				9
(iii)	$4 = 2 + A + B$	M1	Condition on x	
	$17 = 7 + 5A + B$	M1	Condition on y	
	$A = 2, B = 0$	M1	Solve	
	$x = 2 + 2\mathrm{e}^{-3t}$	F1	Follow their GS	
	$y = 7 + 10\mathrm{e}^{-3t}$	F1	Follow their GS	
		B1	Sketch of x starts at 4 and decreases	
		B1	Asymptote $x = 2$	
		B1	Sketch of y starts at 17 and decreases	
		B1	Asymptote $y = 7$	
				9