OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4755

Further Concepts For Advanced Mathematics (FP1)

Wednesday **18 JANUARY 2006** Afternoon 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 4 printed pages.

Section A (36 marks)

1 You are given that $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$.

(i) Calculate, where possible, $2\mathbf{B}$, $\mathbf{A} + \mathbf{C}$, $\mathbf{CA}$ and $\mathbf{A} - \mathbf{B}$. [5]

(ii) Show that matrix multiplication is not commutative. [2]

2 **(i)** Given that $z = a + b\mathrm{j}$, express $|z|$ and z^* in terms of a and b. [2]

(ii) Prove that $zz^* - |z|^2 = 0$. [3]

3 Find $\sum_{r=1}^{n} (r+1)(r-1)$, expressing your answer in a fully factorised form. [6]

4 The matrix equation $\begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ represents two simultaneous linear equations in x and y.

(i) Write down the two equations. [2]

(ii) Evaluate the determinant of $\begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix}$.

What does this value tell you about the solution of the equations in part **(i)**? [3]

5 The cubic equation $x^3 + 3x^2 - 7x + 1 = 0$ has roots α, β and γ.

(i) Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. [2]

(ii) Find the cubic equation with roots 2α, 2β and 2γ, simplifying your answer as far as possible. [4]

6 Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$. [7]

Section B (36 marks)

7 A curve has equation $y = \dfrac{3 + x^2}{4 - x^2}$.

(i) Show that y can never be zero. [1]

(ii) Write down the equations of the two vertical asymptotes and the one horizontal asymptote. [3]

(iii) Describe the behaviour of the curve for large positive and large negative values of x, justifying your description. [2]

(iv) Sketch the curve. [3]

(v) Solve the inequality $\dfrac{3 + x^2}{4 - x^2} \leqslant -2$. [4]

8 You are given that the complex number $\alpha = 1 + \mathrm{j}$ satisfies the equation $z^3 + 3z^2 + pz + q = 0$, where p and q are real constants.

(i) Find α^2 and α^3 in the form $a + b\mathrm{j}$. Hence show that $p = -8$ and $q = 10$. [6]

(ii) Find the other two roots of the equation. [3]

(iii) Represent the three roots on an Argand diagram. [2]

[Turn over

9 A transformation T acts on all points in the plane. The image of a general point P is denoted by P′.

P′ always lies on the line $y = 2x$ and has the same y-coordinate as P. This is illustrated in Fig. 9.

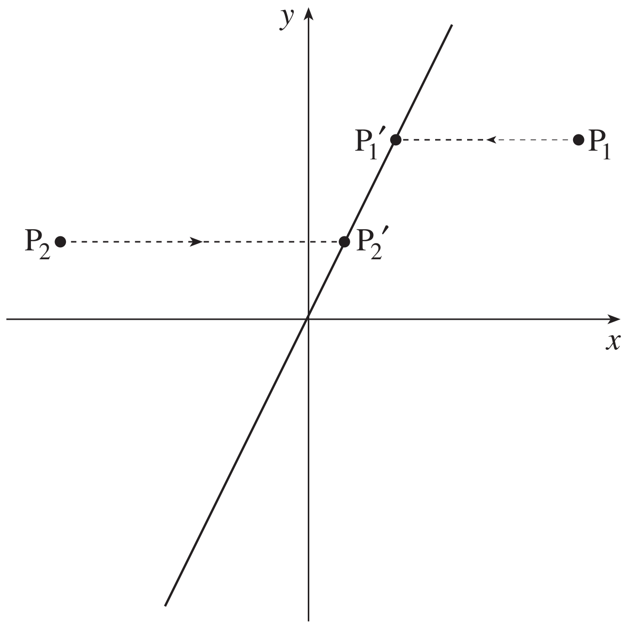


Fig. 9

(i) Write down the image of the point $(10, 50)$ under transformation T. [1]

(ii) P has coordinates (x, y). State the coordinates of P′. [2]

(iii) All points on a particular line l are mapped onto the point $(3, 6)$. Write down the equation of the line l. [1]

(iv) In part **(iii)**, the whole of the line l was mapped by T onto a single point. There are an infinite number of lines which have this property under T. Describe these lines. [1]

(v) For a different set of lines, the transformation T has the same effect as translation parallel to the x-axis. Describe this set of lines. [1]

(vi) Find the 2×2 matrix which represents the transformation. [3]

(vii) Show that this matrix is singular. Relate this result to the transformation. [3]

Mark Scheme 4755
January 2006

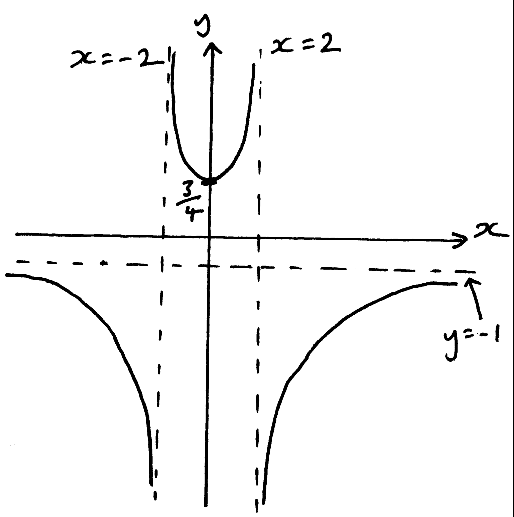
Section A

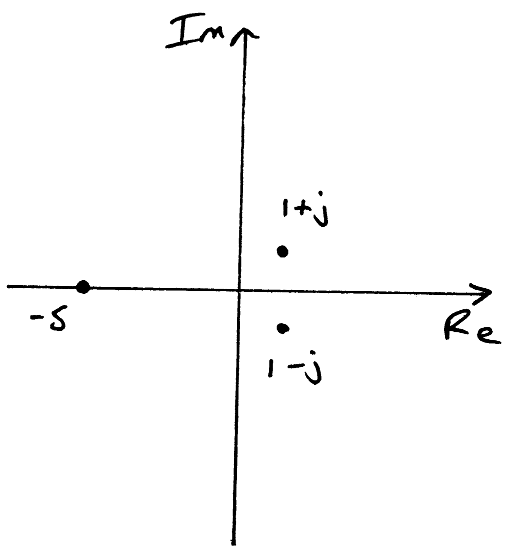
1(i)	$2\mathbf{B} = \begin{pmatrix} 4 & -6 \\ 2 & 8 \end{pmatrix}$, **A + C** is impossible, $\mathbf{CA} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}$, **A - B** = $\begin{pmatrix} 2 & 6 \\ 0 & -2 \end{pmatrix}$	B1 B1 M1, A1 B1 **[5]**	**CA** 3×2 matrix M1
1(ii)	$\mathbf{AB} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 11 & 0 \\ 4 & 5 \end{pmatrix}$ $\mathbf{BA} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 8 & 11 \end{pmatrix}$ $\mathbf{AB} \neq \mathbf{BA}$	M1 E1 **[2]**	Or **AC** impossible, or student's own correct example. Allow M1 even if slip in multiplication Meaning of commutative
2(i)	$\lvert z \rvert = \sqrt{\left(a^2 + b^2\right)}$, $z^* = a - b\text{j}$	B1 B1 **[2]**	
2(ii)	$zz^* = (a+bj)(a-bj) = a^2 + b^2$ $\Rightarrow zz^* - \lvert z \rvert^2 = a^2 + b^2 - \left(a^2 + b^2\right) = 0$	M1 M1 A1 **[3]**	Serious attempt to find zz^*, consistent with their z^* ft their $\lvert z \rvert$ in subtraction All correct
3	$\sum_{r=1}^{n}(r+1)(r-1) = \sum_{r=1}^{n}\left(r^2 - 1\right)$ $= \frac{1}{6}n(n+1)(2n+1) - n$ $= \frac{1}{6}n\left[(n+1)(2n+1) - 6\right]$ $= \frac{1}{6}n\left(2n^2 + 3n - 5\right)$ $= \frac{1}{6}n(2n+5)(n-1)$	M1 M1, A1, A1 M1 A1 **[6]**	Condone missing brackets Attempt to use standard results Each part correct Attempt to collect terms with common denominator c.a.o.

4(i)	$6x - 2y = a$ $-3x + y = b$	B1 B1 **[2]**	
4(ii)	Determinant = 0 The equations have no solutions or infinitely many solutions.	B1 E1 E1 **[3]**	No solution or infinitely many solutions Give E2 for 'no unique solution' s.c. 1: Determinant = 12, allow 'unique solution' B0 E1 E0 s.c. 2: Determinant = $\frac{1}{0}$ give maximum of B0 E1
5(i)	$\alpha + \beta + \gamma = -3,\ \alpha\beta + \beta\gamma + \gamma\alpha = -7,\ \alpha\beta\gamma = -1$	B2 **[2]**	Minus 1 each error to minimum of 0
5(ii)	Coefficients *A*, *B* and *C* $2\alpha + 2\beta + 2\gamma = 2 \times -3 = -6 = \frac{-B}{A}$ $2\alpha \times 2\beta + 2\beta \times 2\gamma + 2\gamma \times 2\alpha = 4 \times -7 = -28 = \frac{C}{A}$ $2\alpha \times 2\beta \times 2\gamma = 8 \times -1 = -8 = \frac{-D}{A}$ $\Rightarrow x^3 + 6x^2 - 28x + 8 = 0$	M1 A3 **[4]**	Attempt to use sums and products of roots ft their coefficients, minus one each error (including '= 0' missing), to minimum of 0
	OR		
	$\omega = 2x \Rightarrow x = \frac{\omega}{2}$ $\left(\frac{\omega}{2}\right)^3 + 3\left(\frac{\omega}{2}\right)^2 - 7\left(\frac{\omega}{2}\right) + 1 = 0$ $\Rightarrow \frac{\omega^3}{8} + \frac{3\omega^2}{4} - \frac{7\omega}{2} + 1 = 0$ $\Rightarrow \omega^3 + 6\omega^2 - 28\omega + 8 = 0$	M1 A1 A1 A1 **[4]**	Attempt at substitution Correct substitution Substitute into cubic (ft) c.a.o.

6	$\sum_{r=1}^{n}\frac{1}{r(r+1)}=\frac{n}{n+1}$ $n = 1$, LHS = RHS = $\frac{1}{2}$ Assume true for $n = k$ Next term is $\frac{1}{(k+1)(k+2)}$ Add to both sides $\text{RHS}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$ $=\frac{k(k+2)+1}{(k+1)(k+2)}$ $=\frac{k^2+2k+1}{(k+1)(k+2)}$ $=\frac{(k+1)^2}{(k+1)(k+2)}$ $=\frac{k+1}{k+2}$ But this is the given result with $k + 1$ replacing k. Therefore if it is true for k it is true for $k + 1$. Since it is true for $k = 1$, it is true for $k = 1, 2, 3$	B1 E1 B1 M1 A1 E1 E1 **[7]**	Assuming true for k (must be explicit) $(k + 1)^{th}$ term seen c.a.o. Add to $\frac{k}{k+1}$ (ft) c.a.o. with correct working True for k, therefore true for $k + 1$ (dependent on $\frac{k+1}{k+2}$ seen) Complee argument

Section A Total: 36

	Section B		
7(i)	$3+x^2 \neq 0$ for any real x.	E1 **[1]**	
7(ii)	$y = -1$, $x=2$, $x=-2$	B1, B1 B1 **[3]**	
7(iii)	Large positive x, $y \to -1^-$ (e.g. consider $x=100$) Large negative x, $y \to -1^-$ (e.g. consider $x=-100$)	M1 B1 **[2]**	Evidence of method required From below on each side c.a.o.
7(iv)	Curve 3 branches correct Asymptotes labelled Intercept labelled  	B1 B1 B1 **[3]**	Consistent with (i) and their (ii), (iii) Consistent with (i) and their (ii), (iii) Labels may be on axes Lose 1 mark if graph not symmetrical May be written in script
7(v)	$\frac{3+x^2}{4-x^2} = -2 \Rightarrow 3+x^2 = -8+2x^2$ $\Rightarrow 11 = x^2$ $\Rightarrow x = (\pm)\sqrt{11}$ From graph, $-\sqrt{11} \le x < -2$ or $2 < x \le \sqrt{11}$	M1 A1 B1 A1 **[4]**	Reasonable attempt to solve Accept $\sqrt{11}$ $x < -2$ and $2 < x$ seen c.a.o.

8(i)	$\alpha^2 = (1+j)^2 = 2j$ $\alpha^3 = (1+j)2j = -2+2j$ $z^3 + 3z^2 + pz + q = 0$ $\Rightarrow 2j - 2 + 3 \times 2j + p(1+j) + q = 0$ $\Rightarrow (8+p)j + p + q - 2 = 0$ $p = -8$ and $p + q - 2 = 0 \Rightarrow q = 10$	M1, A1 A1 M1 M1 A1 **[6]**	Substitute their α^2 and α^3 into cubic Equate real and imaginary parts to 0 Results obtained correctly
8(ii)	$1 - j$ must also be a root. The roots must sum to –3, so the other root is z = -5	B1 M1 A1 **[3]**	Any valid method c.a.o.
8(iii)		B2 **[2]**	Argand diagram with all three roots clearly shown; minus 1 for each error to minimum of 0 ft their real root

Section B (continued)			
9(i)	$(25, 50)$	B1 **[1]**	
9(ii)	$\left(\frac{1}{2}y, y\right)$	B1, B1 **[2]**	
9(iii)	$y = 6$	B1 **[1]**	
9(iv)	All such lines are parallel to the *x*-axis.	B1 **[1]**	Or equivalent
9(v)	All such lines are parallel to $y = 2x$.	B1 **[1]**	Or equivalent
9(vi)	$\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$	B3 **[3]**	Minus 1 each error s.c. Allow 1 for reasonable attempt but incorrect working
9(vii)	$\det\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = 0 \times 1 - 0 \times \frac{1}{2} = 0$ Transformation many to one.	M1 E2 **[3]**	Attempt to show determinant = 0 or other valid argument May be awarded without previous M1 Allow E1 for 'transformation has no inverse' or other partial explanation
			Section B Total: 36
			Total: 72

4755: Further Concepts for Advanced Mathematics (FP1)

General Comments

Most of the candidates for this paper were extremely good, as might be expected for a January sitting of FP1, and consequently it was high scoring.

A small proportion of candidates were not well prepared for a paper of this nature.

The overall standard of scripts was pleasing, although even among the best scripts there was some evidence of immaturity in the written mathematics – missing brackets, imprecise explanations and poor use of notation.

All of the questions worked well, including the slightly out-of-the-ordinary question 9 which involved a singular transformation.

A small number of the weaker candidates either missed out question 9, or made only a very poor attempt, which may indicate they had run out of time.

Comments on Individual Questions

1) **Matrices**

This question was well answered. The most common mistakes resulted from careless arithmetic but a significant number of candidates thought that a 3×2 matrix (on the left) could not be multiplied by a 2×2 matrix (on the right). Another common error was to calculate $2\mathbf{B}$ as $\mathbf{B}^2$.

In part (ii) a large majority of candidates knew the meaning of commutativity but a few muddled it with associativity and some failed to conclude their argument by stating that they had shown, for their chosen **X** and **Y**, that $\mathbf{XY} \neq \mathbf{YX}$.

2) **Complex numbers**

Many candidates got this question fully right but a significant minority did not know the meaning of $|z|$.

Many answers showed poor use of brackets. Many answers were not very logically presented. The most lucid found zz^* and $|z|^2$, then showed the subtraction to give 0.

3) **Series summation**

This question required the use of the standard results for $\sum r^2$ and $\sum 1$. It attracted many correct answers. Nearly all candidates knew how to approach the question but many wrote $\sum 1 = 1$ instead of $\sum 1 = n$. Another common mistake was to fail in the step where both expressions had to be written over a common denominator.

4) **Use of matrices to solve simultaneous equations**

In part (i) candidates had to write a matrix equation as a pair of simultaneous equations and nearly everyone was successful in this.

In part (ii) candidates were required to show that the determinant of the matrix was zero and this was mainly done successfully. They were then asked to interpret this in the context of the equations; only a minority of candidates gave the full answer, that there were either no solutions or infinitely many. A significant minority of candidates attempted to answer this last part in terms of geometrical transformations even though this was not what was asked for.

5) **Roots of a cubic equation**

This question was well answered, with most candidates knowing what to do. However, quite a number of candidates made sign errors.

Part (ii) required candidates to construct a related equation. This could be done either by substitution or by manipulating the roots algebraically to find the coefficients of the new equation. Most, but not all, candidates chose the latter method. A number of candidates gave a polynomial expression rather than equation, missing out the "= 0" and this cost them one mark.

6) **Proof by induction**

Many candidates knew just what to do for this question and scored full marks.

A significant minority, however, did not give the argument explicitly; some used their own phraseology to reduce the amount of writing but in so doing bypassed the essential logic. In extreme cases, candidates scored all the marks for the algebra but none of those for presenting the argument.

The standard of algebra displayed in this question by most candidates was pleasingly high, though many were guilty of poor use of brackets in their working.

A handful of candidates attempted to prove the statement by using methods other than induction and they were given no marks as the question explicitly required proof by induction.

7) **Graph**

This question was on the whole well answered but many candidates dropped a few marks as they went through it and most failed to earn full marks for part (v).

(i) The vast majority of candidates answered this correctly.

(ii) This asked for the equations of the asymptotes and a significant minority of candidates gave the horizontal asymptote as $y = 0$ instead of $y = -1$.

(iii) Candidates were asked to describe the behaviour of the curve for large positive and negative values of x and to justify their answers. Many candidates merely gave the horizontal asymptote, which had already been asked for in the previous part; they were, of course, expected to show, with justification, whether the curve approached this asymptote from above or below.

(iv) Most candidates earned all three marks. However, a significant minority failed to show the asymptotes clearly or give the intercept.

(v) This was about an inequality and candidates were expected to see from their graphs that the solution involved two intervals. However, many made the mistake of giving only one interval. Only the very best candidates got this fully correct. The most successful method was to solve $\frac{3+x^2}{4-x^2}=-2$ and use the solutions, with the sketch, to identify the regions.

8) **Complex numbers**

This question, about a cubic equation with complex roots, was well answered and many candidates got it fully right.

(i) Candidates were asked to justify given values for two of the coefficients of a cubic equation and this caused some difficulty to a minority who failed to equate real and imaginary parts to 0.

(ii) This asked for all the roots of the cubic equation. While most candidates got this right, some used very inefficient methods to do so. The easiest method was to identify the second root as the conjugate of the one that had been given, and then to use either the sum or product of the roots to find the third one. Most candidates used the complex roots to find factors, multiplied to get a quadratic, then divided this into the cubic to get the third factor and hence the root. Many did this correctly, but it was a long-winded method.

(iii) Most candidates got the Argand diagram in part (iii) correct, but there were a few surprising errors with points plotted in quite the wrong positions.

9) **Singular transformation**

Although a few candidates made little or no progress with this question, the majority were able to follow it through part by part and there were many high scores. For those who were on the whole successful the greatest difficulties occurred in part (v), where they had to recognise that any other line with gradient 2 would be translated onto *l*, and in the explanation at the end of part (vii). There were also quite a number of candidates who did not know how to find the matrix in part (vi).

This question demanded some understanding and so it was pleasing to see how many candidates achieved success.

(i) The vast majority of candidates answered this correctly.

(ii) A large proportion got this right although (2*x*, *y*) and $\left(\frac{x}{2}, y\right)$ were common errors.

(iii) Most got this right, but many omitted to answer and y = 2x was a common error.

(iv) Most got this right, but many incorrectly gave lines parallel to the y-axis, rather than the x-axis.

(v) This required some deeper thinking and most candidates got this part wrong.

(vi) A large proportion of candidates did not know how to find the transformation matrix.

(vii) Most who had found a matrix in part (vi) could calculate its determinant and knew the meaning of ‘singular’. Only the very best candidates earned all the explanation marks. The best explanation was to state that the transformation was many-to-one.