OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE **4777/01**

MATHEMATICS (MEI)

Numerical Computation

WEDNESDAY 18 JUNE 2008 Morning

Time: 2 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

CUP/T39242

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- Additional sheets, including computer print-outs, should be fastened securely to the Answer Booklet.

COMPUTING RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities throughout the examination.

INFORMATION FOR CANDIDATES

- The number of marks for each question is given in brackets [ ] at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes. You should note the following points.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.

  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the *formulae* in the cells as well as the *values* in the cells.

  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

1 **(i)** Explain carefully what it means to say that an iteration has first order convergence.
Show that, if y_0, y_1, y_2 are three terms in a first order iteration converging to α, then α may be estimated as $\dfrac{{y_1}^2 - y_0 y_2}{2y_1 - y_0 - y_2}$. [6]

A curve has equation $x^y + y^x = 2$, where $x > 0$ and $y > 0$. Note that the point (1, 1) lies on the curve and that the curve is symmetrical about the line $y = x$. You are given that, for any value of y, there is only one value of x.

(ii) Show that, for $x = 1.1$, the equation may be re-arranged as $y = (2 - 1.1^y)^{\frac{1}{1.1}}$. Set up a spreadsheet to perform the iteration based on this rearrangement. Starting with $y_0 = 1$, obtain y_1 and y_2. Use the result in part **(i)** to obtain a more accurate value of y when $x = 1.1$.

Repeat this process beginning with the more accurate value of y. Comment on the likely accuracy of your new estimate. [6]

(iii) Repeat the process in part **(ii)** to obtain estimates of y for $x = 1.2, 1.3, \ldots 2.0$.
Comment on the likely accuracy of your result for $x = 2$.

Use the spreadsheet to obtain a sketch of the curve for $0 < x \leqslant 2$, $0 < y \leqslant 2$. [12]

2 The trapezium rule, using n strips of width h, is used to find an estimate T_n of the integral

$$I = \int_a^b \mathrm{f}(x)\,\mathrm{d}x,$$

where $b - a = nh$. You may assume that the global error in T_n is of the form

$$A_2 h^2 + A_4 h^4 + A_6 h^6 + \ldots,$$

where the coefficients $A_2, A_4, A_6, \ldots$ are independent of n and h.

(i) Show that $T_n{}^* = \dfrac{4T_{2n} - T_n}{3}$ is an estimate of I with global error of order h^4.

Write down an expression, $T_n{}^{**}$, in terms of $T_{2n}{}^*$ and $T_n{}^*$, that represents an estimate of I with global error of order h^6. [6]

(ii) Use Romberg's method on a spreadsheet to find the value of

$$I = \int_0^2 \frac{x^2}{1 + \mathrm{e}^{-x}}\,\mathrm{d}x$$

correct to 6 decimal places. [9]

(iii) Modify your spreadsheet to find the value of

$$J = \int_0^k \frac{x^2}{1 + \mathrm{e}^{-x}}\,\mathrm{d}x$$

for $k = 0, 0.25, \ldots, 2$. Hence obtain a sketch of J against k. [6]

(iv) Use your spreadsheet to determine, correct to 2 decimal places, the value of k for which $J = 1$. [3]

3 The differential equation

$$\frac{dy}{dx} = 1 - \sqrt{x+y}, \text{ with } y = 0 \text{ when } x = 0,$$

is to be solved numerically.

(i) Use the Runge-Kutta order 4 method with $h = 0.2$ to obtain a sketch of the solution curve for $0 < x < 3$. Give a rough estimate of the coordinates of the turning point (p, q) on the solution curve. Also give a rough estimate of α, the value of x for which the curve crosses the horizontal axis. [11]

(ii) By reducing h appropriately, obtain the values of p, q and α correct to 2 decimal places. [5]

(iii) The differential equation is now generalised to

$$\frac{dy}{dx} = s - \sqrt{x+y}, \text{ with } y = 0 \text{ when } x = 0.$$

Modify your spreadsheet to find, correct to 2 decimal places, the value of s for which $\alpha = 1$. [8]

4 A curve of the form

$$y = a + bx + cx^2 \qquad (1)$$

is to be fitted, using least squares, to a set of data points (x_i, y_i), $i = 1, 2, \ldots, n$.

(i) Show, using partial differentiation, that one of the normal equations is

$$\Sigma y = na + b\,\Sigma x + c\,\Sigma x^2.$$

Write down the other two normal equations. [5]

(ii) Use a spreadsheet to obtain a scatter diagram for the following data.

x_i	0	0.5	1	1.5	2	2.5	3
y_i	1.02	2.08	2.73	3.14	2.87	2.22	1.43

What feature of the data suggests that a curve of the form (1) might be a suitable fit? [3]

(iii) Use a spreadsheet to

(*A*) formulate the normal equations,

(*B*) solve for a, b, c, using Gaussian elimination,

(*C*) find, and comment on, the sum of the residuals,

(*D*) find the residual sum of squares. [16]

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1 (i) Eg: e_{r+1} is approximately ke_r **[E2]**

Uses $y_0 = \alpha + e_0$, $y_1 = \alpha + ke_0$, $y_2 = \alpha + k^2e_0$ or equivalent **[M1A1]**

Convincing algebra to eliminate k hence given result **[A1A1]**

[subtotal 6]

(ii) Convincing re-arrangment **[A1]**

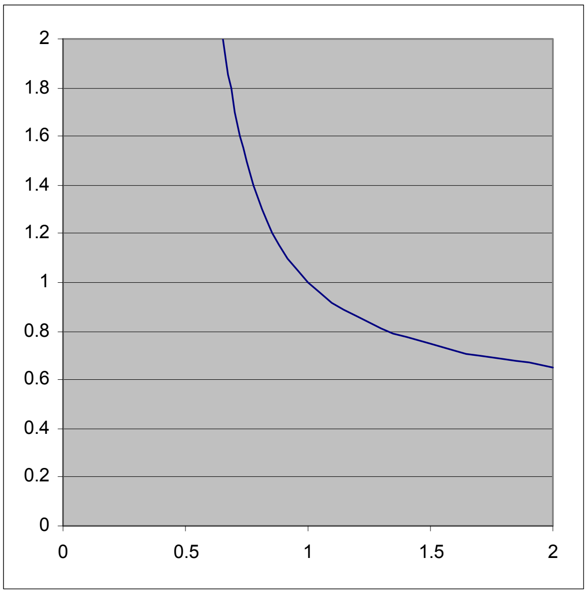
y_0	y_1	y_2	extrap (new y_o)	new y_1	new y_2	extrap		
							once	**[M1A1]**
1	0.908662	0.917409	0.916644	0.916648	0.916647	0.916647	*twice*	**[M1A1]**
					4 or 5 sf looks secure			**[A1]**

[subtotal 6]

(iii)

x	y_0	y_1	y_2	extrap (new y_o)	new y_1	new y_2	extrap	
1.1	1	0.908662	0.917409	0.916644	0.916648	0.916647	0.916647	
1.2	0.916647	0.845937	0.858962	0.856936	0.85695	0.856947	0.856948	**set up**
1.3	0.856948	0.799744	0.815042	0.811814	0.81184	0.811833	0.811835	**SS**
1.4	0.811835	0.763904	0.780556	0.776263	0.776302	0.776288	0.776292	**[M2A2]**
1.5	0.776292	0.734953	0.752555	0.747298	0.747351	0.747329	0.747335	
1.6	0.747335	0.7108	0.729213	0.723043	0.72311	0.723076	0.723087	**values**
1.7	0.723087	0.690112	0.70934	0.702258	0.70234	0.702292	0.70231	**[A3]**
1.8	0.70231	0.671996	0.692131	0.684095	0.684194	0.684128	0.684155	
1.9	0.684155	0.655831	0.677026	0.667954	0.668075	0.667985	0.668023	
2	0.668023	0.641175	0.663627	0.653402	0.65355	0.653427	0.653483	
					3 or 4 sf looks secure			**[A1]**

x	y
0.653483	2
0.668023	1.9
0.684155	1.8
0.70231	1.7
0.723087	1.6
0.747335	1.5
0.776292	1.4
0.811835	1.3
0.856948	1.2
0.916647	1.1
1	1
1.1	0.916647
1.2	0.856948
1.3	0.811835
1.4	0.776292
1.5	0.747335
1.6	0.723087
1.7	0.70231
1.8	0.684155
1.9	0.668023
2	0.653483



organise data [M1A1]

graph G2

Sub Total 12

TOTAL 24

2 (i) $T_n - I = A_2h^2 + A_4h^4 + A_6h^6 + \ldots$

$T_{2n} - I = A_2(h/2)^2 + A_4(h/2)^4 + A_6(h/2)^6 + \ldots$	**[M1A1]**
$4(T_{2n} - I) - (T_n - I) = b_4h^4 + b_6h^6 + \ldots$	**[M1]**
$4T_{2n} - T_n - 3I = b_4h^4 + b_6h^6 + \ldots$	**[A1]**
$(4T_{2n} - T_n)/3 - I = B_4h^4 + B_6h^6 + \ldots$	**[A1]**
($T_n^* = (4T_{2n} - T_n)/3$ has error of order h^4 as given)	
$T_n^{**} = (16T_{2n}^* - T_n^*)/15$ has error of order h^6	**[B1]**
	[subtotal 6]

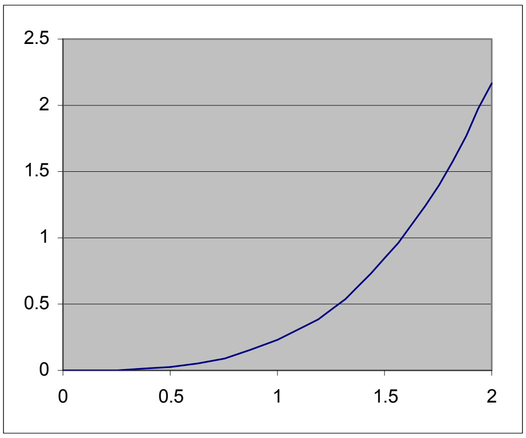
(ii)

x	f(x)	T	T*	T**	(T***)		
0	0						
2	3.523188	3.523188					
1	0.731059	2.492653	2.149141				
0.5	0.155615					f:	**[A1]**
1.5	1.839543	2.243905	2.160989	2.161779			
0.25	0.035136					*T:*	**[M1A2]**
0.75	0.382038						
1.25	1.214531					*T*:*	**[M1A1]**
1.75	2.609105	2.182155	2.161572	2.161611	2.161608		
0.125	0.0083					*T**:*	**[M1A1]**
0.375	0.083344						
0.625	0.254435					*answer:*	**[A1]**
0.875	0.540367						
1.125	0.955439						
1.375	1.509072						
1.625	2.206199						
1.875	3.048173	2.166744	2.161606	2.161609	2.161609		

[subtotal 9]

(iii)

k	I
0	0
0.25	0.002847
0.5	0.024686
0.75	0.089495
1	0.225935
1.25	0.466242
1.5	0.845007
1.75	1.398068
2	2.161609



modify SS **[M2]**

values of I **[A2]**

graph **[G2]**

[subtotal 6]

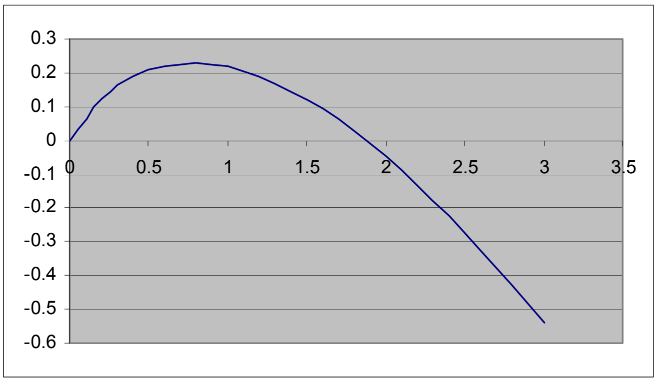
(iv)

k	I			
1.57	0.980739	accept 1.57	*evidence of t&e:*	**[M2]**
1.58	1.001291	or 1.58	*result:*	**[A1]**
1.579	0.999223	(or in between)		

[subtotal 3]
[TOTAL 24]

3 (i)

h	x	y	k 1	k 2	k 3	k 4	
0.2	0	0	0.2	0.110557	0.121189	0.086653	
0.2	0.2	0.125024	0.085978	0.063177	0.064854	0.046393	
0.2	0.4	0.189763	0.046408	0.031125	0.032033	0.018694	
0.2	0.6	0.221666	0.018708	0.007021	0.007628	-0.00291	**setup**
0.2	0.8	0.229182	-0.0029	-0.01239	-0.01194	-0.02066	**[M3]**
0.2	1	0.217146	-0.02065	-0.02863	-0.02828	-0.0357	
0.2	1.2	0.188783	-0.03569	-0.04256	-0.04228	-0.04872	**values**
0.2	1.4	0.146433	-0.04871	-0.05472	-0.05449	-0.06015	**[A3]**
0.2	1.6	0.091887	-0.06015	-0.06547	-0.06527	-0.07031	
0.2	1.8	0.026567	-0.0703	-0.07506	-0.07488	-0.07941	
0.2	2	-0.04836	-0.0794	-0.08369	-0.08353	-0.08762	
0.2	2.2	-0.13194	-0.08761	-0.0915	-0.09136	-0.09507	
0.2	2.4	-0.22334	-0.09507	-0.0986	-0.09849	-0.10187	
0.2	2.6	-0.32186	-0.10187	-0.1051	-0.105	-0.1081	
0.2	2.8	-0.42689	-0.1081	-0.11107	-0.11097	-0.11382	
0.2	3	-0.53789	-0.11382	-0.11656	-0.11647	-0.1191	



[G2]

Maximum about (0.8, 0.23) root about 1.8 **[A1A1A1]**

[subtotal11]

(ii) Eg: **[M2]**

h = 0.01 gives (p, q) as (0.77, 0.22743) hence (0.77, 0.23) **[A1A1]**

h = 0.01 gives root as between 1.87 and 1.88 accept either **[A1]**

[subtotal5]

(iii) Eg:

s	h	x	y	k 1	k 2	k 3	k 4	
1	0.01	0	0	0.01	0.009	0.009025	0.008621	
1	0.01	0.01	0.009112	0.008618	0.008314	0.008319	0.008065	
1	0.01	0.02	0.017437	0.008065	0.007844	0.007847	0.007649	**Mods**
1	0.01	0.03	0.025286	0.007649	0.007468	0.00747	0.007303	**[M3]**
1	0.01	0.04	0.032757	0.007303	0.007147	0.007148	0.007002	**t & e**
								[M3]

s = 0.715, h = 0.01 gives root closest to x = 1 accept 0.71 to 0.72 **[A2]**

[subtotal8]

[TOTAL 24]

4 (i) $Q = \Sigma (y - a - bx - cx^2)^2$ **[M1]**

dQ/da = 0 gives $\Sigma y = na + b \Sigma x + c \Sigma x^2$ as given **[M1A1]**

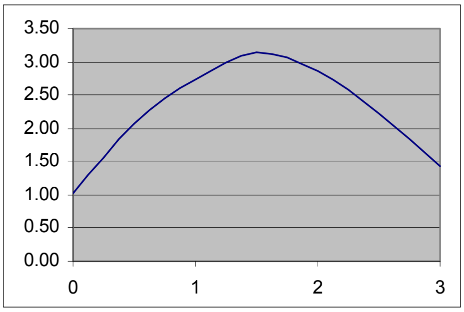
other equations: $\Sigma xy = a \Sigma x + b \Sigma x^2 + c \Sigma x^3$ **[B1]**

$\Sigma x^2y = a \Sigma x^2 + b \Sigma x^3 + c \Sigma x^4$ **[B1]**

[subtotal 5]

(ii)

X	Y
0	1.02
0.5	2.08
1	2.73
1.5	3.14
2	2.87
2.5	2.22
3	1.43



[G2]

roughly parabolic (quadratic) in shape **[E1]**

[subtotal 3]

(iii)

x	y	xy	x^2y	x^2	x^3	x^4	
0	1.02	0	0	0	0	0	
0.5	2.08	1.04	0.52	0.25	0.125	0.0625	
1	2.73	2.73	2.73	1	1	1	
1.5	3.14	4.71	7.065	2.25	3.375	5.0625	
2	2.87	5.74	11.48	4	8	16	
2.5	2.22	5.55	13.875	6.25	15.625	39.0625	
3	1.43	4.29	12.87	9	27	81	**[M2]**
10.5	**15.49**	**24.06**	**48.54**	**22.75**	**55.125**	**142.1875**	**[A2]**

normal equations:

7	10.5	22.75	15.49		**form equations**
10.5	22.75	55.125	24.06		**[M1A1]**
22.75	55.125	142.1875	48.54	a= 1.017619	**solution**
	-6.46154	-21	0.554615	b= 2.562143	**[M2A2]**
	-2.69231	-10.5	1.656923		
		-1.75	1.425833	c= -0.81476	

x	y	y fitted	residual	res^2
0	1.02	1.017619	0.002381	5.67E-06
0.5	2.08	2.095	-0.015	0.000225
1	2.73	2.765	-0.035	0.001225
1.5	3.14	3.027619	0.112381	0.012629
2	2.87	2.882857	-0.01286	0.000165
2.5	2.22	2.330714	-0.11071	0.012258
3	1.43	1.37119	0.05881	0.003459
			-3.6E-15	**0.029967**

y fitted [M1A1]

residuals [M1A1]

residual sum is zero (except for rounding errors) as it should be **[E1]**

residual sum of squares is 0.029967 **[A1]**

[subtotal 16]

[TOTAL 24]

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General Comments

The candidature for this paper was, once again, small. Most of the candidates seemed well prepared for the paper and some scored very highly.

Comments on individual questions

1) **Solution of an equation; acceleration**

   There was some fudging of the algebra in the first part and nobody scored full marks, but most candidates produced a respectable score.

2) **Romberg's method**

   This was the least popular question, though it was well done by those who attempted it.

3) **Runge-Kutta method**

   All the candidates who tackled this question new what to do, but some made slips in transferring the new value of *y* from one row to the next in the spreadsheet.

4) **Least squares curve fitting**

   Again, all the candidates understood what to do, but there were some errors in solving the equations.