

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

**Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education**

MEI STRUCTURED MATHEMATICS

4776

Numerical Methods

Monday **19 JUNE 2006** Morning 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

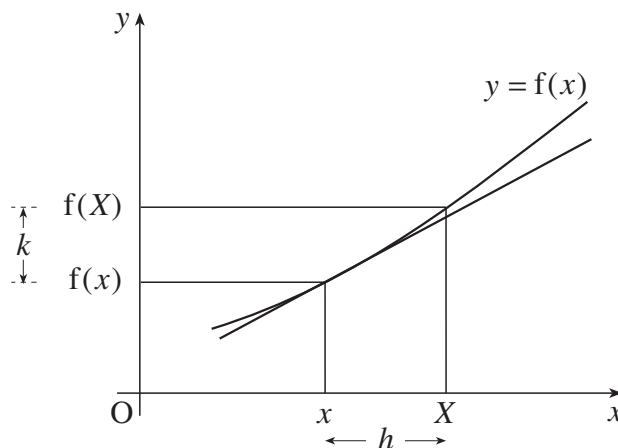
INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 4 printed pages.

Section A (36 marks)

- 1 In the diagram below, X is an approximation to x with error h . Also $f(X)$ is an approximation to $f(x)$ with error k . Show, by considering the tangent at x , that $k \approx hf'(x)$.



Use this result, with $f(x) = \sqrt{x}$, to estimate the maximum possible error in \sqrt{X} when $X = 2.5$ correct to 1 decimal place. [7]

- 2 Show that the equation

$$x^5 - 5x + 1 = 0$$

has a root in the interval $[0, 1]$.

Apply two iterations of the false position method to obtain an estimate of this root. Give your answer to 3 decimal places.

Determine whether or not your answer is correct to 3 decimal places. [8]

- 3 The integral $I = \int_0^2 \sqrt{1+2^x} dx$ is to be found numerically.

Obtain the estimates given by the mid-point rule and the trapezium rule with $h = 2$. Use these values to obtain a Simpson's rule estimate of I .

Given that the mid-point rule estimate with $h = 1$ is 3.510 411, obtain as efficiently as possible a second trapezium rule estimate and a second Simpson's rule estimate.

Give the value of I to the accuracy that appears justified. [8]

- 4 Given the data in the table below, find three estimates of $f'(2)$.

h	0	0.1	0.01	0.001
$f(2+h)$	1.4427	1.3478	1.4324	1.4416

Discuss briefly the likely accuracy of these estimates.

[6]

- 5 Show, by means of a difference table, that the function $g(x)$ tabulated below is approximately but not exactly quadratic.

x	1	2	3	4	5	6
$g(x)$	3.2	12.8	28.4	50.2	77.9	111.6

Use Newton's forward difference formula to estimate the value of $g(1.5)$.

[7]

Section B (36 marks)

- 6 (i) Show that the equation

$$x^2 = \tan x \quad (*)$$

(where x is in radians) has a root in the interval $[4.6, 4.7]$.

Use the bisection method with starting values 4.6 and 4.7 to find this root with maximum possible error 0.0125. [9]

- (ii) You are now given that equation (*) also has a root in the interval $[7.7, 7.9]$. Show that 7.7 and 7.9 are *not* suitable starting points for the bisection method. Explain with the aid of a sketch graph how this situation arises. [5]

- (iii) Using only the fact that equation (*) has a root in the interval $[7.7, 7.9]$, write down the best possible estimate of the root. Determine whether or not this estimate is correct to 1 decimal place. [4]

7 The following values of the function $f(x)$ are known.

x	1	2	4
$f(x)$	-3	8	36

It is required to estimate $D = f'(2)$ and $I = \int_1^4 f(x) dx$.

(i) Use the forward difference method to estimate D .

Use the trapezium rule to obtain the best possible estimate of I . [4]

(ii) Use Lagrange's method to find the quadratic that passes through the given points.

Hence find new estimates of D and I . [11]

(iii) Comment on the extent to which the estimates in part (i) agree with those in part (ii). [3]

Mark Scheme 4776
June 2006

1 Sketch, with explanation. [M1A1E1]

$f'(x) = 1/(2\sqrt{x})$ [M1A1]

hence mpe is approx $0.05 / (2\sqrt{2.5}) = \frac{0.01581}{1} (0.016)$ [M1A1]

(or $0.05 / 2\sqrt{2.45} = \frac{0.01597}{2}$ or $0.05 / 2\sqrt{2.55} = \frac{0.01565}{6}$)

[TOTAL 7]

2

	x	0	1						
	f(x)	1	-3	change of sign so root					[M1A1]
	a	b	f(a)	f(b)	x	f(x)			
	0	1	1	-3	0.25	-0.24902			[M1A1]
	0	0.25	1	-0.24902	0.20015				
					6	-0.00046			[A1]
					0.200 to 3 dp				[A1]
	x	0.1995	0.2005						
	f(x)	0.00281	-0.00218	sign change so root is correct to 3 dp					[M1A1]

[TOTAL 8]

3

	h	M	T	S				
		3.46410	3.65028	3.52616				
	2	2	2	2				values
		3.51041	3.55719	3.52600				
	1	1	2	4				

[A1A1A1A1A1]
evidence of efficient formulae for T and S
[M1M1]
[A1]

3.526(0) appears to be justified

[TOTAL 8]

4

	h	0	0.1	0.01	0.001	
	f(2 + h)	1.4427	1.3478	1.4324	1.4416	
	est f'(2)		-0.949	-1.03	-1.1	

[M1A1A1A1A1]

Clear loss of significant figures as h is reduced [E1]

Impossible to know which estimate is most accurate [E1]

[TOTAL 6]

5

	x	g(x)	Δg	Δ ² g	
	1	3.2	9.6	6	
	2	12.8	15.6	6.2	second differences nearly constant
	3	28.4	21.8	5.9	so approximately quadratic
	4	50.2	27.7	6	
	5	77.9	33.7		
	6	111.6			

table
[M1A1]
[E1]

$$g(1.5) = 3.2 + 0.5 \cdot 9.6 + 0.5 \cdot (-0.5) \cdot 6/2 = 7.25$$

[M1A1A1A1]

[TOTAL 7]

NB: $3\pi/2 = 4.71$ (not reqd)

6 (i)	x	4.6	4.7						
	$x^2 - \tan(x)$	12.2998	3	-58.6228	change of sign, so root				[M1A1]
	a	b	sign f(a)	sign f(b)	x	sign f(x)	mpe		
	4.6	4.7	1	-1	4.65	1	0.05		[M1A1]
	4.65	4.7	1	-1	4.675	-1	0.025		[M1A1]
	4.65	4.675	1	-1	4.6625		0.012		[M1A1]
					root is 4.6625 with mpe		5		[A1]
					0.0125				[subtotal 9]

(ii)	x	7.7	7.9						
	$x^2 - \tan(x)$	52.8471	3	84.1251	1	no change of sign, so no evidence of root			[M1A1]
	Sketch showing asymptote for $\tan(x)$ at $5\pi/2 = 7.854$								[G2]
	So x^2 curve is above $\tan(x)$ at both end points								[E1]
									[subtotal 5]

(iii)	best possible estimate is 7.8								[A1]
	x	7.75	7.85						[M1]
	$x^2 - \tan(x)$	50.4801	-189.529	change of sign so 7.8 is correct to 1 dp					[A1E1]
									[subtotal 4]

[TOTAL 18]

7									
(i)	$D = (36 - 8) / (4 - 2) = 14$								[M1A1]
	$I = 0.5 (-3 + 8) + (8 + 36) = 46.5$								[M1A1]
									[subtotal 4]
(ii)	$q(x) = -3(x-2)(x-4)/(1-2)(1-4) + 8(x-1)(x-4)/(2-1)(2-4) + 36(x-1)(x-2)/(4-1)(4-2)$								[M1A1A1A1]
	$= -(x^2 - 6x + 8) - 4(x^2 - 5x + 4) + 6(x^2 - 3x + 2)$								[A1]
	$= x^2 + 8x - 12$								[A1]
	$q'(x) = 2x + 8$ so $D = 12$								[M1A1]
	$\int q(x) dx = x^3/3 + 4x^2 - 12x$ so $I = 45$								[M1A1A1]
									[subtotal 11]
(iii)	Large relative difference between estimates of D								[E1]
	Small relative difference in estimates of I								[E1]
	To be expected as integration is a more stable process than differentiation								[E1]
									[subtotal 3]

[TOTAL 18]

4776 - Numerical Methods (Written Examination)

General Comments

Once again, candidates varied greatly in their level of preparation for this examination. The best showed an admirable grasp of the techniques and approaches required in numerical mathematics. Some, however, had little knowledge of even the most elementary material. Candidates generally were happier with the calculations than with any analysis or interpretation of results.

Comments on Individual Questions

1) **Error analysis**

This question proved to be beyond the grasp of the majority of candidates. Only a handful could produce the simple argument that the gradient, $f'(x)$ is approximately equal to k/h . The requirement to use the given result was ignored by most. No credit could be given for any other approach to the error in the function.

2) **Solution of an equation**

This question was tackled far more successfully, though some candidates used methods other than that required, such as the secant method or the Newton-Raphson method. In part (ii) a substantial minority did not appreciate the need to show a change of sign in the interval (0.1995, 0.2005). Simply showing that $f(0.200)$ is very near to zero does not establish that 0.200 is correct to 3 d.p., and it scores no marks.

3) **Numerical integration**

Most candidates did this question well, though some lost marks by working with too few decimal places. The relationships between the methods were better understood than in recent examinations. The final answer can be justified as 3.256 or 3.2560; most, but by no means all, candidates with the correct working gave one or other of these values.

4) **Numerical differentiation**

The three estimates were obtained correctly by almost all candidates, but the comments on their likely accuracy often missed the point. Numerical differentiation with very small h , as in this case, involves the subtraction of nearly equal numbers and so gives rise to a progressive loss of precision.

5) **Newton's forward difference method**

The majority understood that, as the second differences are almost constant, the data can be well approximated by a quadratic. Estimating $g(1.5)$ caused little difficulty, though some who had the correct formula substituted erroneous values into it.

6) **Location of roots, solution of an equation**

Identifying the root in part (i) of this question was found easy, but many then carried out the bisection process inefficiently, often performing several iterations more than required. A substantial number of candidates did not work with the maximum possible error and so had no clear idea of when to stop. Some assumed that the maximum possible error at the start of the process would be 0.5.

In part (ii), most could show that there is no change of sign in f between $x = 7.7$ and 7.9 . The explanations of why this might be were generally poor. Often it was stated that there must be two (or more) roots. Very few correctly identified an asymptote as being responsible.

In part (iii), the point estimate of 7.8 was obtained by almost everyone, but many did

not then correctly determine whether 7.8 is correct to 1 d.p. Many simply argued that the root could be anywhere in the interval (7.7, 7.9) so that 7.8 was not secure to 1 d.p.

7)

Numerical differentiation and integration, errors

The value of D was generally correct, but the value of I was often based on a single interval rather than on two. The majority of candidates handled Lagrange's formula correctly, though in some centres it appeared to be unknown. Further estimates of D and I were usually obtained correctly from the polynomial. Comparisons between the estimates were generally rather vague and uninformative. Candidates were required to make comments to the effect that though the absolute differences in the values of D and I were of the same order, the relative difference in the values of I was less than that for D . It was rare to see any such analysis.