

**ADVANCED SUBSIDIARY GCE
MATHEMATICS (MEI)**

4776/01

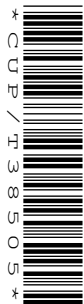
Numerical Methods

THURSDAY 24 JANUARY 2008

Morning

Time: 1 hour 30 minutes

Additional materials: Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)



INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks for each question is given in brackets [] at the end of each question or part question.
- The total number of marks for this paper is 72.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (36 marks)

- 1 The equation $f(x) = 0$ is known to have a single root. Given that $f(2) = 0.24$ and $f(3) = 0.03$, use the secant method to obtain an estimate of the root. Show, by means of a sketch, that this estimate could be very inaccurate. [5]

- 2 For the integral $I = \int_0^1 \frac{2 - \sqrt{x}}{2 + \sqrt{x}} dx$, find the values given by
- (A) the trapezium rule with $h = 1$,
- (B) the mid-point rule with $h = 1$.

Use these two values to obtain a further trapezium rule estimate and a Simpson's rule estimate of the integral. [8]

- 3 The function $f(x)$ has the values shown in the table.

x	0	1	3
$f(x)$	2.00	2.57	3.85

Use Lagrange's method to find the estimate of $f(2)$ given by fitting a quadratic function to the data. [7]

- 4 Show that the equation $x^3(2 - x) = 1$ has a root in the interval $(1.5, 2)$. Use the bisection method to find the root with maximum possible error 0.0625. [6]

Determine how many further iterations of the bisection process would be required to reduce the maximum possible error to less than 0.005. [2]

- 5 A numerical derivative is being found using the forward difference approximation. Show, by means of a sketch, that a large value of h may lead to a large error. [3]

The function $g(x)$ has the values shown in the table correct to 3 decimal places.

x	2	2.001	2.01	2.1
$g(x)$	3.610	3.612	3.633	3.849

Obtain three estimates of the derivative of the function at $x = 2$. Use your answers to show that, in numerical differentiation, a smaller value of h may not always lead to greater accuracy. [5]

Section B (36 marks)

6 The function $f(x)$ has the values shown in the table.

x	3	4	5	6
$f(x)$	1	3	-1	-10

(i) Use Newton's forward difference interpolation formula to fit a quadratic to the points at $x = 3, 4, 5$. Use this quadratic to estimate

(A) the value of x at which $f(x)$ takes its maximum value,

(B) the value of x in the interval $(4, 5)$ for which $f(x) = 0$.

Show that the quadratic does not pass through the fourth data point. [12]

(ii) Use Newton's forward difference interpolation formula to estimate $f(4.5)$ using a cubic. (Note that you are not required to find the cubic in terms of x .)

Hence obtain a Simpson's rule estimate of $\int_3^6 f(x) dx$. [6]

[Question 7 is printed overleaf.]

- 7 (i) The number 2.506 628 is known to be correct to 6 decimal places. Write down the maximum possible error and calculate the maximum possible relative error. [3]
- (ii) A computer adds up 1000 numbers each of which has been *rounded* to 6 decimal places. Calculate the maximum possible error in the sum. Explain why an error of this magnitude is unlikely to arise in practice. [3]
- (iii) A computer adds up 1000 numbers each of which has been *chopped* to 6 decimal places. Calculate the maximum possible error in the sum. What is the most likely error in practice? Explain your answer. [5]
- (iv) A computer program in which numbers are rounded to 7 significant figures is used to sum the following numbers. All intermediate answers used in calculations are rounded to 7 significant figures.

1, 0.000 000 1, 0.000 000 2, 0.000 000 3, 0.000 000 4.

Find the answers the program will give if the numbers are summed

- (A) from left to right,
 (B) from right to left.

Explain the difference in the two answers. [3]

- (v) A simple computer program is written to find the following sum.

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{1000^3}.$$

The answer obtained is 1.202 051. When the terms are summed in reverse order the answer is 1.202 056. State, with an explanation, which of these is likely to be more accurate.

When the same two calculations are performed on a spreadsheet the two answers that are displayed are identical. What *two* features of a spreadsheet explain why this happens? [4]

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1	x	2	3	root = $(2 \times 0.03 - 3 \times 0.24) / (0.03 - 0.24)$	[M1A1]
	f(x)	0.24	0.03	= 3.142857	[A1]
	Eg: graph showing turning point at x = 3 with root some way to the left or the right.				[G2]
[TOTAL 5]					

2	x	f(x)			
	0	1			
	1	0.333333	T1 =	0.666667	[M1A1]
	0.5	0.477592	M =	0.477592	[M1A1]
			hence	T2 = (T1 + M)/2 =	0.572129 [M1A1]
			and	S = (T1 + 2*M)/3 =	0.540617 [M1A1]
[TOTAL 8]					

3	x	0	1	3	
	f(x)	2	2.57	3.85	
					3 terms: [M1]
					form: [M1]
					use x=2: [M1]
	f(2) =	$2(2-1)(2-3)/(0-1)(0-3) + 2.57(2-0)(2-3)/(1-0)(1-3) + 3.85(2-0)(2-1)/(3-0)(3-1)$			[A1A1A1]
	=	3.186667	(3.19)		[A1]
[TOTAL 7]					

4	x	1.5	2		
	$x^3(2-x)-1$	0.6875	-1	change of sign, so root (<i>may be implied</i>)	[M1A1]
	a	b	x	$x^3(2-x)-1$	mpe
	1.5	2	1.75	0.339844	0.25 [M1A1]
	1.75	2	1.875	-0.17603	0.125 [A1]
	1.75	1.875	1.8125		0.0625 [A1]
	4 further iterations reqd: mpe 0.0325, 0.015625, 0.0078125, 0.00390625				[M1A1]
[TOTAL 8]					

5	Sketch showing curve, tangent, chord, h. Makes clear that tangent and chord have substantially different gradients.				[G3]
	h	0	0.1	0.01	0.001
	g(2 + h)	3.61	3.849	3.633	3.612
	est g'(2)		2.39	2.3	2
	Clear loss of significant figures as h is reduced				[E1]
[TOTAL 8]					

6	(i)	<table border="0" style="width: 100%;"> <thead> <tr> <th style="text-align: left;">x</th> <th style="text-align: left;">f(x)</th> <th style="text-align: left;">Δf</th> <th style="text-align: left;">$\Delta^2 f$</th> <th style="text-align: left;">$\Delta^3 f$</th> <th></th> </tr> </thead> <tbody> <tr> <td>3</td> <td>1</td> <td></td> <td></td> <td></td> <td></td> </tr> <tr> <td>4</td> <td>3</td> <td>2</td> <td></td> <td></td> <td></td> </tr> <tr> <td>5</td> <td>-1</td> <td>-4</td> <td>-6</td> <td></td> <td></td> </tr> <tr> <td>6</td> <td>-10</td> <td>-9</td> <td>-5</td> <td>1</td> <td></td> </tr> </tbody> </table>	x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$		3	1					4	3	2				5	-1	-4	-6			6	-10	-9	-5	1		[M1A1A1]
x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$																													
3	1																																
4	3	2																															
5	-1	-4	-6																														
6	-10	-9	-5	1																													
		quadratic = $1 + 2(x-3) - 6(x-3)(x-4)/2$			[M1A1]																												
		= $1 + 2x - 6 - 3x^2 + 21x - 36$			[A1]																												
		= $-3x^2 + 23x - 41$			[A1]																												
		$q'(x) = -6x + 23 = 0$ at $x = 23/6$ (= 3.833...)			[M1A1]																												
		$q(x) = 0$ at $x = 4.847(127)$; also at 2.81954 - not reqd.			[M1A1]																												
		$q(6) = -11$ (or point out that the second differences not constant)			[A1]																												
					[subtotal 12]																												
	(ii)	cubic est = $1 + 2(4.5-3) - 6(4.5-3)(4.5-4)/2 + 1(4.5-3)(4.5-4)(4.5-5)/6$ = 1.6875			[M1A1A1] [A1]																												
		$S = 1.5/3 (1 + 4 \times 1.6875 - 10) = -1.125$			[M1A1] [subtotal 6]																												
[TOTAL 18]																																	
<hr/>																																	
7	(i)	mpe 0.000 000 5 mppe 0.000 000 5 / 2.506 628 = 1.99×10^{-7}			[B1] [M1A1] [subtotal 3]																												
	(ii)	mpe 1000 x 0.000 000 5 = 0.000 5 In practice the positive and negative errors will tend to cancel out			[M1A1] [E1] [subtotal 3]																												
	(iii)	mpe 1000 x 0.000 001 = 0.001 In practice 1000 x 0.000 000 5 = 0.000 5 because average error in chopping will be 0.000 000 5			[M1A1] [M1A1] [E1] [subtotal 5]																												
	(iv)	L to R: 1 (or 1.000 000) R to L: 1.000 001 L to R requires 8 sf, (R to L doesn't)			[B1] [B1] [E1] [subtotal 3]																												
	(v)	Reverse order more accurate as that way allows the very small terms at the end of the series to contribute to the sum.			[E1] [E1]																												
		The spreadsheet is likely to work to greater accuracy			[E1]																												
		The spreadsheet works to more sf than are displayed			[E1]																												
[subtotal 4]																																	
[TOTAL 18]																																	

4776: Numerical Methods (Written Examination)

General Comments

It is pleasing to be able to report that many candidates performed well on this paper, showing a good grasp of theory and technical facility. However, a weakness that has been commented on before was still evident: some candidates present their work as a jumble of figures rather than in tabular form. Numerical algorithms lend themselves to a systematic style of presentation that aids understanding, checking and, indeed, marking. Those adopting a more casual approach are likely to make more errors and to score fewer method marks.

Comments on individual questions

1) Secant method

The single step of the secant method was generally carried out accurately, though some candidates gave the answer to only one decimal place. The sketch to show that the method could be very inaccurate was not well done. Indeed, many sketches showed cases where the method would be accurate. The sketch could have shown a turning point on the curve close to $x = 3$, with the root some considerable distance away.

2) Integration

This question was very well done with many scoring full marks. The vast majority understood clearly what was meant by trapezium and mid-point estimates with $h = 1$. Then almost as many were able to generate the required further estimates by using these values.

3) Lagrange's interpolation method

Most candidates gained high marks on this question with relatively few committing the error of confusing the x and the $f(x)$ values. The question asked for the value of $f(2)$. There was no need, therefore, for candidates to find and simplify the quadratic in its algebraic form.

4) Bisection method

Candidates were all familiar with the idea of bisection, though many chose not to lay out their work in a table. In some cases this made it impossible for the examiner to decide which value the candidate was offering as the answer. The final section, identifying the number of further iterations required, was very well done.

5) Differentiation

The sketch to show that the forward difference can be very inaccurate when h is large was often poorly done. Most candidates failed to show clearly that the chord from x to $x + h$ could have a very different gradient to the tangent at x . Sometimes the point illustrated was that $f(x)$ and $f(x + h)$ could be very different. This is true, but not relevant. A large difference between $f(x)$ and $f(x + h)$ might arise when the gradient of the chord is, in fact, very close to the gradient of the tangent.

The numerical part of the question, however, was done well with the majority of candidates appreciating that as h decreases precision is lost.

6) **Newton's interpolation method**

A significant minority made errors in drawing up the difference table. Some simply made numerical slips, while others consistently had their differences the wrong way round. (The latter, though unconventional, is acceptable if the signs are handled correctly in subsequent work.) The quadratic, required in algebraic form, was usually correct, though some made errors in the simplification. Finding the maximum was just a simple matter of differentiation and finding the root just required the use of the quadratic formula. Some candidates invented very elaborate methods here. In the final part, the cubic estimate of $f(4.5)$ does not require the cubic to be found algebraically. The integral comes out as a negative number. Only a very small number of candidates supposed that they were finding an area and that the answer should be positive.

7) **Errors and representation of numbers**

Though many candidates made a good attempt at this question, it was clear that some had a poor understanding of how computers store and process numbers. Rounding was understood better than chopping. A common error was to assume that the maximum possible error in chopping to 6 decimal places is 0.000 000 9 rather than 0.000 001. Most understood that the two sums in part (iv) will produce different answers, but a significant minority missed the point altogether. Curiously, what is essentially the same phenomenon in part (v) was better understood. In the final section the two required features of a spreadsheet were that it works to greater accuracy – i.e. a greater number of significant figures – than the simple computer program, and that it does not display all the significant figures that it uses. The second point was appreciated by only a few. The existence of “guard digits” in computer software and in calculators is an important point – and it is quite distinct from working to a large number of significant figures.