ADVANCED SUBSIDIARY GCE

MATHEMATICS (MEI)

Numerical Methods

4776/01

Candidates answer on the Answer Booklet

OCR Supplied Materials:

- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:

- Scientific or graphical calculator

Monday 24 May 2010
Afternoon

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Section A (36 marks)

1 **(i)** Show that the equation

$$\frac{1}{x} = 3 - x^2 \qquad (*)$$

has a root, α, between $x = 1$ and $x = 2$.

Show that the iteration

$$x_{r+1} = \frac{1}{3 - x_r^2},$$

with $x_0 = 1.5$, converges, but not to α. **[5]**

(ii) By rearranging (*), find another iteration that does converge to α. You should demonstrate the convergence by carrying out several steps of the iteration. **[3]**

2 A function f(x) has the values shown in the table.

x	2.8	3	3.2
f(x)	0.9508	0.9854	0.9996

(i) Taking the values of f(x) to be exact, use the forward difference method and the central difference method to find two estimates of f′(3). State which of these you would expect to be more accurate. **[5]**

(ii) Now suppose that the values of f(x) have been rounded to the four significant figures shown. Find, for each method used in part **(i)**, the largest possible value it gives for the estimate of f′(3). **[2]**

3 **(i)** X is an approximation to the number x such that $X = x(1 + r)$. State what r represents.

Show that, provided r is small, $X^n \approx x^n(1 + nr)$. **[4]**

(ii) The number $G = 0.577$ is an approximation to the number g. G is about 0.04% smaller than g. State, in similar terms, relationships between

(*A*) G^2 and g^2,

(*B*) $\sqrt{G}$ and $\sqrt{g}$. **[3]**

4 The expression, $\sin x + \tan x$, where x is in radians, can be approximated by $2x$ for values of x close to zero.

(i) Find the absolute and relative errors in this approximation when $x = 0.2$ and $x = 0.1$. **[4]**

(ii) A better approximation is $\sin x + \tan x \approx 2\left(x + \frac{x^3}{k}\right)$, where k is an integer.

Use your results from part **(i)** to estimate k. **[3]**

5 A quadratic function, $f(x)$, is to be determined from the values shown in the table.

x	1	3	6
$f(x)$	–10	–12	30

Explain why Newton's forward difference formula would not be useful in this case.

Use Lagrange's interpolation formula to find $f(x)$ in the form $ax^2 + bx + c$. **[7]**

Section B (36 marks)

6 The integral

$$I = \int_1^{1.8} \sqrt{x^3 + 1}\, dx$$

is to be estimated numerically. You are given that, correct to 6 decimal places, the mid-point rule estimate with $h = 0.8$ is 1.547 953 and that the trapezium rule estimate with $h = 0.8$ is 1.611 209.

(i) Find the mid-point rule and trapezium rule estimates with $h = 0.4$ and $h = 0.2$.

Hence find three Simpson's rule estimates of I. **[7]**

(ii) Write down, with a reason, the value of I to the accuracy that appears to be justified. **[2]**

(iii) Taking your answer in part **(ii)** to be exact, show in a table the errors in the mid-point rule and trapezium rule estimates of I.

Explain what these errors show about

(*A*) the relative accuracy of the mid-point rule and the trapezium rule,

(*B*) the rates of convergence of the mid-point rule and the trapezium rule. **[8]**

7 **(i)** Show that the equation

$$x^5 - 8x + 5 = 0 \qquad (*)$$

has a root in the interval (0, 1).

Find this root, using the Newton-Raphson method, correct to 6 significant figures.

Show, by considering the differences between successive iterates, that the convergence of the Newton-Raphson iteration is faster than first order. **[11]**

(ii) You are now given that equation (*) has a root in the interval (1.4, 1.5). Find this root, correct to 3 significant figures, using the secant method. Determine whether or not the secant method is faster than first order. **[8]**

OCR
RECOGNISING ACHIEVEMENT

GCE

Mathematics (MEI)

Advanced GCE **4776**

Numerical Methods

Mark Scheme for June 2010

1(i)

x	LHS		RHS	
1	1	<	2	(Change of sign implies root.)
2	0.5	>	-1	*(or equivalent)*

[M1A1]

r	0	1	2	3	4	5	6
x_r	1.5	1.333333	0.818182	0.429078	0.355127	0.347961	0.347352

[M1A1]

State or clearly imply convergence outside the interval (1, 2) **[E1]**

(ii) E.g. $x_{r+1} = \sqrt{(3 - 1/x)}$

r	0	1	2	3	4	5
x_r	1.5	1.527525	1.531452	1.532	1.532077	1.532087

E.g. $x_{r+1} = 3/x - 1/x^2$ **[B1]**

r	0	1	2	3	4	5
x_r	1.5	1.555556	1.515306	1.544287	1.523326	1.538438

[M1A1]

[TOTAL 8]

2(i) Forward difference: $(0.9996 - 0.9854)/0.2 = 0.071$ **[M1A1]**

Central difference: $(0.9996 - 0.9508)/0.4 = 0.122$ **[M1A1]**

Central difference expected to be more accurate. **[E1]**

(ii) Forward difference maximum: $(0.99965 - 0.98535)/0.2 = 0.0715$ **[B1]**

Central difference maximum: $(0.99965 - 0.95075)/0.4 = 0.12225$ **[B1]**

[TOTAL 7]

3(i) r is the relative error (in X as an approximation to x) **[E1]**

$X^n = x^n (1 + r)^n$ $(1 + r)^n = 1 + nr$ (provided r is small) **[M1M1A1]**

(ii) G^2 (= 0.332 929, not required) is about 0.08% smaller than g^2

$\sqrt{G}$ (= 0.795 605, not required) is about 0.02% smaller than $\sqrt{g}$ **[M1A1A1]**

[TOTAL 7]

4(i)

x	sin + tan	$2x$	error	rel error	*accept:*	+ve, +ve
0.2	0.401379	0.4	-0.00138	-0.00344		*-ve, +ve*
0.1	0.200168	0.2	-0.00017	-0.00084		*-ve, -ve*

[M1A1A1A1]

(ii) $2 \times 0.2^3 / k = 0.00138$ gives $k = 11.59$ Either of these (or other methods) **[M1A1]**

$2 \times 0.1^3 / k = 0.00017$ gives $k = 11.76$ to suggest $k = 12$ **[B1]**

[TOTAL 7]

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5 Data not equally spaced in x **[E1]**

$f(x) = -10(x - 3)(x - 6) / (1 - 3)(1 - 6) - 12(x - 1)(x - 6) / (3 - 1)(3 - 6) + 30(x - 1)(x - 3) / (6 - 1)(6 - 3)$

[M1A1A1A1]

$f(x) = -(x^2 - 9x + 18) + 2(x^2 - 7x + 6) + 2(x^2 - 4x + 3)$ **[A1]**

$= 3x^2 - 13x$ **[A1]**

[TOTAL 7]

6(i)

h	*M*	*T*	*S*		
0.8	1.547953	1.611209	**1.569038**	*M:*	**[M1A1A1]**
0.4	**1.563639**	**1.579581**	**1.568953**	*T:*	**[M1A1]**
0.2	**1.567619**	**1.571610**	**1.568949**	*S:*	**[M1A1]**

[subtotal 7]

(ii) 1.56895 appears justified Comparison of last two S values, e.g.: **[B1]**
last change in *S* is -0.000004; next change negligible **[E1]**

[subtotal 2]

(iii)

h	*M* error	*T* error		
0.8	-0.02100	0.04226	*accept consistent*	
0.4	-0.00531	0.01063	*use of other sign*	
0.2	-0.00133	0.00266	*convention*	**[M1A1A1]**

(A) *M* errors are about half the *T* errors so *M* is twice as accurate as *T* **[E1A1]**
(B) Errors for both *T* and *M* reduce by a factor of 4 as h is halved so **[E1]**
the rates of convergence are the same, both second order **[A1A1]**

[subtotal 8]
[TOTAL 17]

7(i) f(0) = 5, f(1) = -2. (Change of sign implies root.) **[M1A1]**

f '(*x*) = $5x^4 - 8$ hence N-R formula **[M1A1]**

r	0	1	2	3	4	
x_r	0.5	0.634146	0.638232	0.638238	0.638238	**[M1A1A1]**
differences		0.134146	0.004086	5.98E-06	1.29E-11	**[A1]**
ratios			0.030457	0.001462	2.17E-06	**[M1A1]**

The ratios of differences are decreasing (fast) so process is faster than first order **[E1]**

[subtotal 11]

(ii)

r	0	1	2	3	4	
x_r	1.4	1.5	1.458054	1.462741	1.46312	
f(x_r)	-0.82176	0.59375	-0.0747	-0.00559	5.99E-05	**[M1A1A1]**

root is 1.46 correct to 3 sf **[A1]**

	0	1	2	3	4	
differences		0.1	-0.04195	0.004687	0.000379	**[A1]**
ratios			-0.41946	-0.11175	0.080876	**[M1A1]**

The ratios of differences are decreasing (fast) so process is faster than first order **[E1]**
accept 'second order' ***[subtotal 8]***
[TOTAL 19]