Mathematical Problem Solving

A guide for teachers

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Contents

About this guide

This guide has been produced by MEI for teachers of GCSE and A level Mathematics to support them with the problem solving content of these qualifications. The guide, the examples and their solutions are available to download from the MEI website.

Introduction

Both the 2015 9 – 1 GCSE and the 2017 AS and A levels place a strong emphasis on students being able to solve mathematical problems.

Assessment Objective 3 (AO3) in the GCSE is all about *solving problems within mathematics and in other contexts* and states that students should be able to:

- translate problems in mathematical or non-mathematical contexts into a process or a series of mathematical processes
- make and use connections between different parts of mathematics
- interpret results in the context of the given problem
- evaluate methods used and results obtained
- evaluate solutions to identify how they may have been affected by assumptions made

DfE: Mathematics GCSE subject content and assessment objectives 2013

For the higher tier specifications AO3 has a weighting of 30%. At foundation tier this is 25%.

One of the overarching themes specified in the AS and A level content for first teaching from 2017 is OT2: Mathematical Problem Solving. All the AS and A level specifications in mathematics have to require students to demonstrate these skills:

- Recognise the underlying mathematical structure in a situation and simplify and abstract appropriately to enable problems to be solved
- Construct extended arguments to solve problems presented in an unstructured form, including problems in context
- Interpret and communicate solutions in the context of the original problem
- Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy
- Evaluate, including by making reasoned estimates, the accuracy or limitations of solutions], including those obtained using numerical methods
- Understand the concept of a mathematical problem solving cycle, including specifying the problem, collecting information, processing and representing information and interpreting results, which may identify the need to repeat the cycle

- Understand, interpret and extract information from diagrams and construct mathematical diagrams to solve problems, including in mechanics

DfE: Mathematics GCE AS and A level subject content July 2014

It is clear that in these examinations problem solving skills are going to be assessed in a way that is more rigorous than has been done previously. Students will have to answer some questions in an exam in which they will have to choose the appropriate mathematical techniques with little guidance given in the question and there may even be multiple approaches possible for the question. It will be part of a teacher's role to develop the skills in their students to be able to solve these sorts of problems.

For some teachers this will involve a sea change in the way they teach and a number of concerns may immediately spring to mind:

- There won't be enough time to teach problem solving as well as all of the proscribed content; students find problem solving difficult and it lengthens each topic if we include it.
- The examination will mainly be about assessing the main mathematical skills. It's better to concentrate on those as that will get the students through.
- You can't *teach* problem solving skills, some students are just good at it while others are not. Problem solving is particularly difficult for those who have trouble mastering basic techniques, it's better for less able students to concentrate on the basics instead of problem solving.
- I've only just become confident in the techniques at this level. Introducing problem solving will put me at a disadvantage with my students as I have little confidence in this area.

These are all valid concerns and, understandably, teachers can worry about all of them. We can look at each one in turn to see if there is something we can do to at least reduce any fears.

1. There won't be enough time to teach problem solving as well as all of the proscribed content. Students find problem solving difficult and it lengthens each topic if we include it.

   Time constraints have always been a problem in teaching. Educating children is rarely easy and it does take time. We know that the students will have to deal with mathematical problems in an examination and we can be sure that if they have no experience of solving mathematical problems at all, they will be at a real disadvantage. Students have to experience problem solving in order to improve at it, so it has to form part of their education. One of the best ways of doing this is to integrate problem solving into each topic taught. This requires a carefully planned scheme of work that highlights appropriate problems and techniques and places them sensibly in a series of lessons. For example, a problem could be used to introduce a new topic. The students could consider the problem and think about the techniques they need to develop to solve that problem. The teacher could then lead the students through the new mathematical techniques required, including using some routine exercises to develop their skills, before returning to the problem to check their students understanding. Another example would be for a teacher to use a problem to check understanding at an intermediate point in a topic. This could be included in part of an exercise so there are routine questions as well as a problem and could form part of their students' classwork or homework.

2. The examination will mainly be about assessing the main mathematical skills. It's better to concentrate on those as that will get the students through.

   The two subject content documents, *Mathematics GCSE subject content and assessment objectives* and *Mathematics AS and A level content*, provided by the DfE make it clear that each specification must provide for the assessment of problem solving skills. Solely concentrating on procedural skills will not help students cope with the questions that involve aspects of problem solving. It is clearly advantageous for students to see the sorts of problems that may turn up in the examinations. This could involve using problems from the specimen assessment materials and (eventually) past papers regularly in lessons.

3. You can't teach problem solving skills, some students are just good at it while others are not. Problem solving is particularly difficult for those who have trouble mastering basic techniques. It's better for less able students to concentrate on the basics instead of problem solving.

   Whilst it is true that some students seem to be automatically good at solving mathematical problems, it is not true that these skills cannot be taught. Solving problems is about identifying what information you have been given and where you have to get to with that information before plotting a route from one to the other. There are a number of techniques that can be taught to students to enable them to do this. This guide focuses on some of these and explains how they can be integrated into schemes of work.

   For students who find some of the basic skills difficult, providing a reason for using those skills may help them to understand why they are doing something. It could be argued that problem solving is a basic skill in mathematics and that its development may help students to improve other aspects of their mathematics.

4. I've only just become confident in the techniques at this level. Introducing problem solving will put me at a disadvantage with my students as I have little confidence in this area.

   Introducing problem solving can involve a real shift in what mathematics teachers expect to be doing in lessons. Teachers from many other subjects are used to using discussion and group work to encourage their students to develop their thinking skills. They do not always know where their students are going to go with their discussion and have to react accordingly. Integrating problem solving may involve a change in classroom culture. In the NRICH article *Developing a Classroom Culture That Supports a Problem-solving Approach to Mathematics* by Jennie Pennant (2013) which is aimed at primary school teachers, a number of questions are given to help the teachers consider the culture of their classroom:

   - Who does most of the talking in whole-class parts of the lesson?
   - What questions do I ask?
   - Who answers the questions?
   - How well do I listen to the students' answers and seek to understand what they are saying?
   - What do I do with the students' answers?

- How do I facilitate the learning?
- How confident are the students to take a risk, to try out ideas, to make mistakes?
- What does my body language communicate?

The teacher's role in developing problem solving skills is more as a facilitator rather than the fount of all knowledge. Considering the questions above when planning and delivering lessons including problem solving is a good start for a less confident teacher. This guide includes a number of suggestions for including problem solving in lessons and provides some practical activities that teachers can adapt so they are able to introduce problem solving techniques to their students and lead those students through problems to completion. After trying these ideas out, confidence should improve to the extent that teachers will feel able to plan and try out their own ideas for developing problem solving skills.

Section 1: Asking the right questions

Starting off

When experienced problem-solvers encounter a new problem, a typical first step after reading it is to ask themselves a number of questions to identify the key features of the problem and to get an idea of what the problem is about. These questions are usually asked *internally*. The ability to ask internal questions is not a skill that most students start out with, it has to be developed. Most people, when asked to think about a problem will simply let their mind wander without having any real focus. Asking appropriate questions is a key technique for students when solving problems as it helps to provide the focus needed.

A good set of questions will allow a student to identify:

- The information that they can see immediately
- The information they think they will need
- The mathematical skills that they will need to use

One way of developing this important skill is to give students the chance to ask a series of *external* questions when solving a problem. A student who is used to asking a number of *external* questions will find the transition to asking themselves questions much easier and will have a better focus when solving mathematical problems.

There is a very straightforward way to incorporate the development of this important skill in everyday lessons that also allows for the development of procedural skills. The technique is to present students with a problem in which some key information is missing. The students should spend some time considering the problem and thinking about what information is missing before asking the teacher for answers to their questions.

There are 4 examples below. The first two (one at AS/A level and one at GCSE level) show problems that are incomplete and the teacher needs to be questioned in order for students to get that information. Examples 3 and 4 show problems in which all of the required information is given but the students have to find it themselves. In this case, the questioning technique is used to help the students find out what they need to know. Each example is followed by a commentary showing the questions that students asked when the problems were trialled with a variety of classes. The level of each class and where the content lies in either the subject content for the 2017 AS and A levels or the 2015 GCSE subject content and assessment objectives is identified at the start of each problem. These problems were originally trialled with students studying for older variants of GCSE and AS/A level. They have been carefully selected so they can be used with the newer qualifications.

Example 1 – AS/A level

Solution to example 1

Example 2 – GCSE Higher Tier (calculator required)

Solution to example 2

Progressing with questioning skills

Problems are usually written so they are *complete*; that is they include all of the information needed by the student. Sometimes they will include redundant information that the student will not even use. If a problem can be solved by more than one method, it is often the case that the most efficient method uses the minimum amount of information necessary.

The self-questioning process can be further developed by providing students with problems that contain all of the required information and following a similar process to that used when getting students started with developing their questioning skills.

Example 3 – AS/A Level

Solution to example 3

Example 4 – GCSE

Solution to example 4

Section 2: Using group work

"By giving our students practice in talking with others, we give them frames for thinking on their own."

Lev S. Vygotsky. Mind in Society: Development of Higher Psychological Processes. 1978

Encouraging students to work with their peers and involve themselves in discussion when solving problems can be a very powerful way for them to develop their problem solving skills. Group work forms a part of the experience of many students through their school life and is a key feature for lessons in some subjects. Mathematics is often the exception to this rule, with some teachers feeling less than confident in using group work effectively. There are a number of texts that deal with using group work effectively that give detailed advice for setting up groups to avoid dominance by a single student, designing activities and organising classrooms to accommodate group work. This section looks at two examples of types of activities that involve students working in groups or pairs.

Problem Solving Mysteries

Problem solving mysteries involve students piecing together the information they need to solve a problem from a series of clues. All of the students in a group solving one of these mysteries has to have some involvement and has to make some value judgements on the quality of the information they hold. In each of the examples below (one for AS/A level and the other for GCSE), the initial instructions for the activity are given as well as the cards that each group of students is supplied with. Each example is followed by a commentary based on students using the activity.

Example 5 – AS/A level

Solution to example 5

Example 6 – GCSE

Solution to example 6

Working in pairs:
Improving explanation, sharing thinking process and reflecting on methods

One particularly effective form of paired work encourages students to pass on good problem solving ideas at the same time as improving their explanation and working. The activity involves giving each person in the pair a different problem to solve. The problems should be of the same type in terms of skills and methods but should not be interrelated. The problems used should be in two to three parts with both problems having the same number of parts.

1. Instructions

- Each student in the pair should attempt the first part of their problem individually.
    - There is a set time limit for this (from 5 – 10 minutes depending on the problems set).
    - It does not matter if any student does not finish their part of the problem within the time given.
    - Students who do finish within the time limit should not go on to the second part of the problem, they should consider how they have set their working out and how they might explain what they have done to their partner.
- Once the time is up, each person should explain what they have done and why, going through their working to whatever point they reached.
    - Each student should be given the same time to do this (5 minutes)
    - After each explanation, the student who has been listening has 2 additional minutes to ask any questions about that explanation.
- The students then swap problems and either complete the first part of their new problem before moving on to the second part (if the first part was not completed) or just attempt the second part of the question (if the first part was completed).
    - There is a set time limit for this (5 – 10 minutes)
    - It does not matter if the problem is not completed
    - If the student does complete the problem, they should think about their working and explanation as before.
- The students explain their working as they did before
- Unfinished problems can then be examined as a class to collect together any new ideas or insights
- Solutions to problems that can't be solved by the class are modelled by the teacher

The following example illustrates this type of activity being used in an AS/A level classroom.

Example 7 – AS/A level

Solution to example 7

Example 8 - GCSE

Solution to example 8

Section 3: Key problem solving skills

The problem solving process

George Pólya's book *How to Solve It*, originally published in 1945, has been so useful in helping students develop their problem solving skills that it is still in print today. It contains some excellent advice and examines the process of mathematical problem solving in detail. In particular, Pólya provides a four step procedure for solving mathematical problems:

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Reflect on what has been done

The phrases "devise a plan" and "carry out the plan" can be slightly deceptive. It can be hard even for experienced problem solvers to have a hard-and-fast methodology for problem solving. The plan usually involves sketching out some ideas and then trying to tie them together. It is perhaps better to paraphrase Pólya's procedure as:

1. Understand the problem
2. Engage with the problem
3. Work towards the solution
4. Reflect on what has been done

Any mathematician approaching a new problem will go through some form of this procedure in order to solve it. In preparing students for problem solving, it is important for a teacher to share these steps and look in detail at the strategies that can be used at each stage.

1. Understand the problem

The first part of the process is perhaps the hardest; many students will not "understand the problem" straight away so they should be encouraged to employ some strategies so that they can start to see what the problem is getting at.

Students should go in to problem solving understanding the following:

- They will not be set a problem for which they have not encountered the necessary skills.
- They do know quite a lot of useful mathematics. The problems are set to bring that out.
- Teachers/examiners are not trying to catch students out. They are providing the chance to show how good they can be.
- Mathematical problems often try to hint at a greater mathematical "truth". If they can work out what that is, they will have a much better chance of solving the problem.
- There are often clues to what can be done to solve the problem in the way it is worded.
- Ultimately (i.e. in the final examination), nobody is going to lead a student by the hand through a problem. They have to find their own path to the solution.

An important point for a teacher to get across early on is that doing something is almost always better than doing nothing! It is often the case that a student will look at a problem and simply stop. They need some strategies to get them past the stage of "I haven't got a clue what this is going on about" as quickly as possible. The strategies they use should enable a student to get some sort of feel for the problem they will be trying to solve. Doing something to begin solving a problem will focus a student's mind and hopefully start to give them some insight into what the problem is about and how they might solve it.

1. Techniques for getting started

The first thing that students should do with any problem is to spend a few minutes reading the problem thoroughly from start to finish. They can then try to answer these questions:

- Do the final stages of the problem indicate where the problem is trying to take you?
- Are there some clear "target" expressions you have to find?
- Does the form of what you have to find (what it looks like) indicate the skills that you need to use?

This reflection stage encourages students to think first and try to identify the clues in the problem. A key part of this is reading the problem through to the end. The required end result is often key in identifying the processes needed to solve the problem.

The next stage is for students to gather together all of the information that they can glean from the problem. They should be encouraged to note this down as a list or on a diagram. Again, there are questions that the students should be considering whilst doing this:

- Does this fit in with any skills you know from A level/GCSE?
- Have you included all of the information that has been given?
- Why you have been given a particular bit of information?
    - Is it important that the solutions are whole numbers?
    - Is it important that $a \neq b$?

- Where the problem setter is trying to get you to go in terms of testing your skills?

Hopefully, by this stage, the students will have at least a general idea of what the problem is trying to get them to do and they can start to consider the mathematical skills they will need to employ. It is a good idea for students to note down all of the skills and mathematical “tricks” that they think may be useful. They should be encouraged to focus on what will be needed for the problem. If something is clearly not relevant they shouldn’t list it.

As part of the understanding stage there are two more things that students could do to help them to make progress. These are to draw a diagram and to express the problem in their own words. Diagrams and sketch graphs can often draw a student’s attention to the key features of a problem. They should be big enough for a student to add information to. They are not, however, the be all and end all in problem solving. At their most effective, diagrams and sketch graphs can unlock the solution to a seemingly intractable problem. At their worst they can muddy the waters and add to a student’s confusion. Diagrams should be used where they are appropriate and students should always reflect on the value of the diagram they have chosen to draw. In a similar way, expressing the problem in a student’s own words can often focus their mind on what the problem wants them to do. Again, students should only use this technique when appropriate. When starting out with problem solving, students should experiment to find out when it is helpful to do this and when it simply wastes time.

2. Engage with the problem

When starting out with problem solving, it is important that students are not afraid to try things out. The engagement stage is where they will get their initial ideas together and start to develop a path through the problem. It should be a time where the teacher reminds the students that doing something is better than doing nothing. With some problems, the process that students need to go through is so unclear that all they can do is try something out. Hopefully, their early attempts to understand the problem will have given them some ideas of things they could try. They should be encouraged to experiment at this stage as even if they don’t get anywhere with what they initially try, they may well get some new insight into the problem.

Techniques for the engagement stage

Students could plan to do a variety of things depending on the problems set. Some of the more usual things that students can plan to do are:

- **Look for a pattern**

  Sequences can be very useful and can help in a number of situations. Students should plan to break a problem down into stages that start at something small and then build up. They should think about how to get from one stage to the next.

- **Look for a visual representation**

  Diagrams can be a very powerful way of getting key bits of information together. Students should consider what sort of diagram could help. Should they draw a sketch graph? Would representing the situation as a network shed any light?

- **Work backwards**

  For problems with results to aim for or when a student has an idea of what the solution should look like, they can plan their steps backwards from there. Their plan can include the sort of things they would have to do to get the final result. They can ask the question "can I see a path through to the solution from the starting conditions?

- **Consider some cases**

  Sometimes students can make good progress by trying out some numbers, formulae or expressions. They can be quite focused in how they do this by:

  - Starting small and building up a bit at a time. This is related to looking for a pattern.
  - Looking at some extreme cases. They should ask themselves "what happens if I make this value very small/large?"
  - Looking at some critical cases e.g. if they have a factor $(x-4)$, they might investigate what happens as x approaches 4 from either direction.

- **Find another way of thinking about the problem**

  If it doesn't look like there is an obvious way to start the problem, students should try to think of other ways to approach it. For example if a problem is about the number of successes and the combinations of these it may be easier and more direct to consider the number of failures or if a problem is considering the number of painted faces for a model made of cubes, it may be better to consider the unpainted faces.

- **Think logically**

  This seems easier said than done but it is worth encouraging students to look at a problem and reflect on how they would have to think to solve it. Can they identify the logical steps from one stage to the next?

- **Organise the information**

  Sometimes insight can be gained from putting the information into some sort of table or in a network. This technique can help students to be systematic in how they approach a problem.

- **Solve a simpler related problem**

  If a problem is initially very complex it is sometimes useful for students to think about a simpler related problem and solve that. The method for solving the simpler problem may well give insight about the original problem.

- **List (or imagine a list of) all of the possibilities**

  Sometimes by attempting to make an exhaustive list of all of the possibilities in a problem a student can begin to see how that problem works.

- **Make an educated guess and test this out**

  This seems like the most tenuous of techniques but sometimes a student will just have a feel for what they have to do to solve a particular problem. Students should not be afraid to go with this sort of fuzzy forward thinking and see where it leads. It may be a blind alley but they may well learn more about the problem in doing this.

3. Work towards the solution

Usually students have actually started working on the solution to a problem during the engagement stage. It is important that they employ one key technique whilst they are doing this and that is to reflect on what they are doing so that they know when to abandon a technique and move on to something different. It is very easy for students to fall in to the trap of "loving" the mathematics that they are producing. It always feels good to string some mathematical processes together and this can give an illusion of progress. The tough thing is having to be brutal when that mathematics is going nowhere. It is vitally important for students to reflect on what they are achieving when they are running through calculations. They should be encouraged to regularly ask themselves "Where are these calculations taking me?" and "What will I find out by doing this?" As soon as they know a technique is not going anywhere useful they should think "Have I learned anything about what I should have been doing?" and then abandon it and try something else.

4. Reflect on what has been done

This part of the process is the most neglected. There are some key questions that students should be asking themselves:

- Have I actually solved the problem that was set?
- Have I completely solved the problem that was set?
  - Is there anything I've missed?
  - Have I taken all cases into account?
  - Have I done anything to reduce the number of solutions (such as inadvertently dividing by 0)?
  - Have I used all of the information I was given in the problem? If they haven't, students may want to check that the information they haven't used was redundant in the first place.
- Are there any holes in my argument?
  - Could someone pick my argument apart with a lot of "What if…?" questions?

There is one final and very important question that students should be asking themselves once they have completed any problem and that is:

- What have I learned about problem solving in doing this?

Problem solving skills are improved by experience of solving problems. Each problem solved should add to that pool of experience so students should reflect on what they have been doing, what was successful, any "tricks" they've picked up and what they would and wouldn't do again when solving similar problems.

Helping students remember problem solving techniques

It can be difficult for students to remember all of the problems solving techniques described above and to a certain extent it would be better if the use of those techniques is instinctive rather having students follow a list of what to try. To help students remember these techniques it is a good idea to have the steps of problem solving process and the techniques that relate to it as part of a classroom display. Students tackling problems in class can then glance at the displays when they are unsure of what to try at any stage of the problem. A series of posters has been prepared to accompany this guide.

Section 4: Helping students to complete problems

"… a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking."

Pólya, George. *How To Solve It*. 1945.

In order for students to successfully develop their problem solving skills, they actually have to have experience of solving problems from start to finish. For many students this can be a daunting and challenging task. They may be able to see how to start working on a problem but may get bogged down or lose heart when they don't seem to be making any progress with solving it. Resilience is an important attribute in problem solving but it is unlikely that most students will arrive with the ability to persevere when things get difficult. Like other problem solving skills, resilience has to be developed in students and it will only be developed if students feel they are achieving some sort of success. Students have to feel that solving a problem is worth their effort and that they will get there if they keep going. It is the teacher's role to make sure that the students experience some success in their problem solving and therefore feel that they are able to tackle other problems as a result of this.

In the quote above, Pólya highlights some key features that teachers should include in their lessons. Challenging the students' curiosity is important; students who think a problem is worth solving will make much more effort to solve that problem. Using problems that are proportionate to students' knowledge is key to developing problem solving skills. Students need to be able to actually solve the problems set so those problems must be based around skills that are already embedded. Problem selection is incredibly important when developing students' resilience. A poor set of problems that the students have no hope of solving can do more harm than good. Pólya includes the teacher as a key part in the process, in challenging curiosity, in selecting suitable problems and finally in helping students to solve their problems with stimulating questions. This last part of the trinity can be quite challenging for teachers. It is very tempting for a teacher when helping a struggling student to say "you do it like this" and solve the problem for them. The challenge is to help the student by asking them questions that get them thinking so that they actually can solve the problem themselves.

Initial prompts and ideas

Part of the process of getting students to complete problems is embedded in the techniques already encountered in this guide. The initial stage of the students reading through a problem and then coming up with their questions is an excellent platform for providing early prompts to get them started. Key questions that the students ask should be written clearly so that all in the class can see them. Additional prompts based on these can be given to help the class get started.

Asking stimulating questions and giving helpful prompts

In the same way that students have to develop their problem solving skills, teachers should constantly develop their prompting and questioning skills. This can only be done if the teacher actually practices whilst the students are attempting to solve problems. Lessons involving problem solving can be a learning exercise for all concerned.

One skill a teacher needs to develop is the ability to identify the path a student is taking through a problem so that they can react to any difficulties the student has immediately and appropriately.

An important first question a teacher can ask a student who is struggling with a problem is “can you explain what you have done so far?” This sometimes provides a student with the opportunity to reflect on what they are thinking or have already done which may help them to see the way forward themselves. It is more typical that a student will genuinely be struggling to make progress and some direct help is needed. The following examples show possible questions and prompts that can be used to help students through some specific problems:

Example 9 – AS/A Level

Solution to example 9

Example 10 – GCSE Higher Tier

Solution to example 10

Further problems along with suitable hints and prompts can be found in the examples book accompanying this guide.

Offering strategic hints

As students become more confident in their general mathematical skills, the nature of the hints and prompts given by a teacher should change from the more technical hints about the skills to use to more strategic hints about problem solving. These hints should be based on the techniques encountered in Section 3: Key Problem Solving Skills.

Below is a summary of the questions, hints and prompts that can be given at each stage of the problem solving process.

1. Understand the problem

Look at the whole of the problem. Is there anything about what you have to find in the final stages that indicate where the problem is trying to take you?

Do you have to give the solution in a particular form?
Does that form indicate the skills that you need to use to solve the problem?

Are there some clear “target” expressions you have to find?
Are there any signposts that you are going in the right direction?

Can you see what mathematical skills you will be using yet?
Will there be related skills from the same area of mathematics that may come in handy?
Are there any mathematical “tricks” that might come in to play here?

Have you found all of the information that has been given?
Can you see any information that has been “disguised” by the way the problem has been presented?

Why you have been given this particular bit of information?
What does this information tell you about the problem?

What skills (apart from problem solving) do you think the problem is designed to test?

2. Engage with the problem

Is this problem about looking for a pattern?
Can you break it down into stages that start at something small and then build up?
How will you get from one stage to the next?

Is there a useful diagram or sketch you could draw?
Can the problem be expressed geometrically?
Would a sketch graph help?
Would a network help?

Is it possible to work backwards from what the problem wants you to find?
Can you see a path through the problem from the starting conditions given?

Would it be useful to consider some specific cases?
Are there any numbers you could try out?
Do you have a rough idea of a formula that you could try out?
Is it wort testing the extreme cases?
What is the smallest this could be?
What is the largest?
Are there any critical values that are worth looking at?

Is there another way of looking at this problem?
Is it easier to find the opposite of what you are looking for and adapt that solution?
Is there a way of representing the problem differently?

Can you see a logical way of approaching this problem?
Are there some clear steps that should be followed?

Is there a good way to organise the information given?
Do you need to redraft what you've found to make it clearer for yourself?

Is there a simpler version of this problem that you could solve?
How have you simplified the problem?
Can you put the detail back in?

Can you list all of the possibilities?
How would you go about listing all of the possibilities?

Can you make an educated guess about the method to use?
Do you see where this is roughly heading?

3. Work towards the solution

Where does you work seem to be taking you?

Do you feel like you are getting anywhere?

Does your method seem to be working?

How much work have you put into this so far?
Does it seem to be producing results?
Do you think you will get to the solution in a sensible time?

What will you find out by doing that?

Are you going down a dead end there?
Should you abandon that approach and try something else?
Have you learned anything useful about what you should have been doing?

4. Reflect on what has been done

Have you actually solved the problem that was set?
Have you completely solved the problem that was set?
Is there anything you've missed?
Have you taken every case into account?
Have you done anything to reduce the number of solutions (such as inadvertently dividing by 0)?
Have you used all of the information you were given in the problem?
Was that information redundant or do you need to take it into account?
Have you used the best method?

Could you shorten your method by being more efficient?

- with your calculations
- with your reasoning?

Are there any holes in your argument?
Could I pick your argument apart by asking a lot of "What if…?" questions?

What have you learned about problem solving in doing this?
Are there any useful tips and "tricks" you could use in other problems?

Student review

The final technique in this section is perhaps the most overlooked in typical mathematics lessons and is particularly effective when students are attempting more involved problems or problems in which mathematical modelling features heavily. This technique is to have a mid-session review of where students are in their attempts to solve a problem. By some careful questioning, a teacher can allow students to compare their approaches and any successes. This does not need to be done in a particularly involved way with student presentations, it should simply be a review of where certain students are. By carefully selecting which students to ask, the teacher can provide those in the class who are struggling with the problem a chance to reflect on what they are doing and compare their methods to those employed by others.

The teacher's role in this is to select students to report briefly on their approach based on what they have seen whilst the students have been working. They should be prepared to include some methods that are not necessarily going to be successful as it is important that students make a value judgement about their own approach. It is not a problem if a student decides that their method is better than the one being presented. A variety of approaches should be represented as far as it is possible to do so although completely incorrect approaches should be avoided. There should be some essence of mathematical truth in every method reviewed. The teacher should be prepared to carefully rephrase what the students say where necessary and to ask additional probing questions to aid the rest of the class in comparing their methods. At no stage should the teacher say if a particular method is the correct one or not.

For problems that are used year on year, a teacher may store some examples of methods used in previous years and present these to the class.

Section 5: Multiple approaches

"Solving problems in multiple ways contributes to the development of students' creativity and critical thinking."

Leikin, R. and Levav-Waynberg, A. Solution Spaces of Multiple-Solution Connecting Tasks as a Mirror of the Development of Mathematics Teachers' Knowledge. 2008

One key feature of mathematical problems is that it is possible to solve them using a number of different approaches. When developing problem solving skills in students it is important for a teacher to allow them to follow different approaches rather than funnel them towards a specific one. This can create some difficulties for the teacher:

- Standard mark schemes are not helpful
- The teacher has to "think on their feet" to help students
- Students may follow approaches that do not lead to a correct result
- Students may not follow an efficient approach

In order to make things manageable, a teacher can adopt a number of strategies to deal with these difficulties.

1. Standard mark schemes are not helpful

Mark schemes that only allow for one method are not really useful for marking problem solving questions unless those problems are particularly short. It helps to know what the correct final result is but, as it is possible for the routes to vary, there is little value in having individual marks for one particular path identified in detail. Marking of this type is best left to the typical maths questions used to assess specific skills. It is more effective for a teacher to be able to assess the mathematical thinking that has gone into the solution to a problem. Practical techniques for the assessment of problem solving skills need to be developed. This is something that, until recent changes to mathematics education in England, has not been particularly necessary. The 2016 ACME report *Problem solving in mathematics: realising the vision through better assessment* includes the recommendation that

"A range of assessment methods need to be developed and trialled to determine how to assess mathematical problem solving most effectively across phases of education in the future. This work should draw on research methodologies and findings."

Since there are no particularly well developed methodologies for assessing problem solving skills in examinations yet, a classroom teacher is left to find their own way to assess their students' problem solving skills. This can't be done by using detailed mark scheme, instead the teacher should ask him/herself a series of questions to get a feel for the problem solving skills used by each student. This process can be simplified to a certain extent by

- Including a time for student review of some of the techniques as part of the lesson. Students should compare methods and comment on their positives and negatives.

The student review as part of a lesson allows some students to change tack and will result in some standardisation of methods as students who realise that they are following an incorrect or inefficient approach may change tack before submitting their solution.

- Sorting solutions into groups of the same “type” i.e. where the same general approach has been used.

Sorting the solutions into groups of the same type e.g. those that use algebra, those that use sketches etc. should make it easier to compare how skills have been employed. Comparing a few examples of similar methods will usually make it easier to identify the deficiencies in a student’s method if it is clear that another student has applied the same method more concisely, efficiently, effectively, fluently or accurately. The nature of this type of assessment has to be formative rather than summative and students should be encouraged to reflect on any feedback given.

In assessing students’ problem solving skills, teachers should ask themselves the following questions

- Has the student found the correct answer? A correct result accompanied by some sensible looking calculations implies that a correct method has been followed. The teacher can then look for clear steps that link the situation to the answer to check the student’s reasoning.
- Has the student presented their solution in clear steps?
- Has the student been efficient? How many stages were required before the solution was reached? How did this compare to the methods used by other students?
- How easy is it to find a general solution from the student’s method?

The following examples one at A level and one at GCSE each show two different approaches to the same problem and how each may be assessed.

Example 11 - AS/A Level

Solution to example 11

Example 12 – GCSE level

Solution to example 12

2. The teacher has to “think on their feet” to help students

When using short, topic-based problems that have been integrated into a series of lessons to assess understanding as well as problem solving there are usually few variations in the method applied. This makes helping students through a problem a relatively straightforward procedure. The difficulty arises when a teacher wishes to use a more open-ended problem to assess their students’ problem solving skills. Some careful planning can mitigate this to a certain extent and reduce the amount of “thinking on their feet” that a teacher has to do. In selecting the problem the teacher should look at the route through to its solution and think carefully about any other possible methods that could work. In introducing the problem, the initial question and answer session will automatically result in some students adopting a common approach. A carefully placed review of approaches being used around the classroom can also have the same effect. It is only when a student is determined to follow their own unique method (and there is nothing wrong with them doing this) that a teacher will really have to think quickly. It is often difficult for a teacher to immediately see what a student is attempting to do when solving a problem by their own method. A key part of this process is to talk to the student about what they are thinking and doing. In order to help a student with whichever method they are using, a teacher should first help the student to talk their ideas through. This has at least two useful effects; the student has to think about their method and what exactly they are trying to do with it and the teacher has a chance to identify the method that the student has been trying to use. When starting to introduce problems with multiple routes to their solutions, the learning curve for a teacher is quite steep. With experience, the ability to help students following different approaches becomes more natural, particularly when problems have been used in previous years and by different teachers and placed into a department's bank of problems with examples of the different approaches that have been used.

3. Students may follow approaches that do not lead to a correct result

It is very easy for anyone using mathematical techniques accurately to be convinced that they are doing the right thing to solve a problem. It is very easy for a student to love the mathematics that they have painstakingly put together. This is fine when the mathematics will lead to a correct solution but there will always be students for whom the approach they are following will lead nowhere useful. The reflection part of the process is therefore essential when students are attempting to solve longer problems. It is important that students have the ability to assess the effectiveness of their methods and the confidence to abandon lines of thought that do not lead to a solution.

A mid-problem review of progress will allow students to:

- reflect on what they have done and if they feel closer to a solution
- compare their approaches to those of others
- reflect on what they have learned about the problem by using their method

There are some key questions that a teacher should be getting their students to consider as a part of this reflection and before they continue

- Is the amount of work you have done sensible for this problem?
  - Should it take that many calculations to solve?
- What do your calculations actually show?
  - What are you trying to do at each stage?
- Can you describe in words what you are trying to do with your method?

These questions when combined with a comparison of methods used around the classroom should help students to assess what they have been doing.

It is important that, at least in the early stages of introducing problem solving, the teacher shares the reasons for having a review of progress part of the way through solving a problem so that the students know that it is possible to be following a method that may not lead to the solution and that they may have to change tack.

4. Students may not follow an efficient approach

The final stage of the problem-solving process should always be to reflect on what has been done. This can be done very quickly for some problems but for the more involved ones where the teacher is encouraging development of solid problem solving skills, students should spend some time assessing the methods they have used. For some students this may involve looking back over their calculations and looking for ways in which they could have been more efficient. The questions that a student should consider at this stage are:

4. Reflect on what has been done

Have you actually solved the problem that was set?
Have you completely solved the problem that was set?
Is there anything you've missed?
Have you taken every case into account?
Have you done anything to reduce the number of solutions (such as inadvertently dividing by 0)?
Have you used all of the information you were given in the problem?
Was that information redundant or do you need to take it into account?
Have you used the best method?
Could you shorten your method by being more efficient?

- with your calculations
- with your reasoning?

Are there any holes in your argument?
Could I pick your argument apart by asking a lot of "What if…?" questions?

What have you learned about problem solving in doing this?
Are there any useful tips and "tricks" you could use in other problems?

This reflection does not necessarily need to be done as part of a lesson. Students could be encouraged to write their solutions out concisely as part of a follow-up homework with annotation to explain what they have done.

Section 6: Integrating problem solving into lessons

"…students who develop conceptual understanding early perform best on procedural knowledge later. Students with good conceptual understanding are able to perform successfully on near-transfer tasks and to develop procedures and skills they have not been taught. Students without conceptual understanding are able to acquire procedural knowledge when the skill is taught, but research suggests that students with low levels of conceptual understanding need more practice in order to acquire procedural knowledge."

Grouws, Douglas A. Cebulla, Kristin J. *Improving student achievement in mathematics*. 2000

Problem solving skills can only be developed if students have the opportunity to actually solve problems. Curriculum and assessment pressure can often lead to the main focus being on straightforward mathematical exercises rather than problems. These are very good at reinforcing skills used in one context but do not give students scope for considering how those skills might be of any use. If the aim of a mathematics teacher is to improve the mathematical understanding and ability of the students in their care, the two parts of learning, developing procedural knowledge and developing conceptual understanding both have to be addressed. Using mathematical problems as part of lessons is a key part of developing conceptual understanding. The problems provide a reason and a need for the procedural knowledge developed through routine exercises. In order to allow for problem-solving skills to be developed, they have to be integrated into lessons so that they are part of the learning process. This can be done by

- Using short topic-based problems as part of exercises for classwork and homework
- Using carefully selected problems as the introduction to a topic
- Including longer, more open-ended problems at carefully selected points through the year
- Using group based activities to aid problem solving discussion

A mathematics department should be confident that their students at all levels are gaining experience of solving problems as an everyday part of what they do.

1. Using short topic-based problems

A mathematics problem does not always need to stretch a student to the limits of their mathematical ability. Short problems based on the topic being studied should allow students the chance to see how a skill may be applied to a situation whilst testing that skill. The following examples show suitable problems at a variety of levels.

Example 13 - GCSE Foundation Tier

Solution to example 13

Example 14 - GCSE Higher Tier

Solution to example 14

Example 15 - AS/A level

Solution to example 15

Example 16 - AS/A level

Solution to example 16

2. Using carefully selected problems as the introduction to a topic

One way to introduce a topic is by providing the students with a short problem that they will eventually solve and asking them to consider what things they will need to be able to do to solve it. These can then be taught and the appropriate skills practised before they go on to solve the initial problem.

The following examples illustrate this for different levels at GCSE and A/AS level.

Example 17 - GCSE Foundation Tier

Solution to example 17

Example 18 - AS/A Level

Solution to example 18

3. Including longer, more open-ended problems at carefully selected points through the year

To develop good problem-solving skills, students need to attempt longer, more-involved problems. These allow for multiple approaches and give students a chance to really think about how they use the mathematical techniques they have been taught and how they may be combined effectively. Longer problems can be used at key points in the year as part of an assessment of how well students are able to think about and apply the skills they have been taught.

It is important for students when tackling these longer problems to be aware that these longer problems are as much about developing problem solving skills as they are about working through the mathematics of the situation. A key element of each lesson in which a more involved problem is set is that the teacher highlights the key problem solving techniques and skills that can be gleaned both initially and part of the way through the session. This, accompanied by student review, reinforces some of the key ideas in the minds of the students.

4. Using group based activities to aid problem solving discussion

Students who are given problems to work through individually and with no discussion will not necessarily develop strong problem solving skills. Discussion of a problem and the thought processes used in solving it is an essential and incredibly powerful way of developing and reinforcing problem solving skills. Students who have the opportunity to exchange information and ideas when solving problems also find that they develop skills that are useful across the board. At its best, peer discussion from undertaking group work can lead to improvements in:

- the way students present their mathematics
- the quality of explanations of working that has been done
- the use of mathematical vocabulary and definitions
- students' understanding of why certain methods work
- the arsenal of "tricks" and techniques available to each student.

Group work can be very effective but needs careful planning so that students maximise their learning of both the mathematical techniques and problem solving skills they are using. When using an activity that involves students working in groups, the teacher has to ensure that:

- the groups are carefully put together to avoid
  - dominance by one student
  - students sitting back and letting others do the work
- there are several opportunities for review so that
  - direction is given to those who are making poor progress
  - ideas can be shared to maximise learning
- the groups are given something that they are capable of solving.

There are various ways to avoid the pitfalls and a great deal of research into using group work effectively has been done. Some subtle "social engineering" when allocating the groups is effective and frequently used by teachers of a variety of subjects. Students can be placed in groups where they have the best chance of getting something out of the activity. The activities used can themselves sometimes help to avoid problems, for example, the problem solving mysteries described in the using group work section of this guide are designed so that every student in a group has to be involved, in making value judgements about the information given, in writing down parts of the solution and in putting the pieces of information together. The paired work activity described also shares these features. Each student has to do something that can be passed on to the next one. They have to think about how they will explain it and what they have written down. It is an activity that demands engagement from all involved.

The final point is one which is dependent on the teacher knowing the capabilities of their students. Any problems chosen should be challenging enough for students to find them worthwhile whilst at the same time being possible for them to complete. For groups experienced at problem solving, a really difficult problem can promote some valuable discussion but students have to be confident in order for this to happen. For less confident students a more pragmatic approach is required; the teacher should select problems that are perhaps at a slightly lower level in terms of mathematical skills than their students have reached so that they are not trying to work at both the top of their mathematical ability as well as the top of their problem solving ability. It is a fine line for the teacher to tread, avoiding underestimating their students whilst at the same time ensuring that they can achieve some form of success.

Section 7: Problem solving in examinations

A level

The Ofqual A Level Mathematics Working Group report on mathematical problem solving, modelling and the use of large data sets in statistics in AS/A level Mathematics and Further Mathematics gives the following examples of the attributes of examination questions that are considered to include problem solving:

A. Tasks have little or no scaffolding: there is little guidance given to the candidate beyond a start point and a finish point. Questions do not explicitly state the mathematical process(es) required for the solution.

B. Tasks provide for multiple representations, such as the use of a sketch or a diagram as well as calculations.

C. The information is not given in mathematical form or in mathematical language; or there is a need for the results to be interpreted or methods evaluated, for example, in a real-world context.

D. Tasks have a variety of techniques that could be used.

E. The solution requires understanding of the processes involved rather than just application of the techniques.

F. The task requires two or more mathematical processes or may require different parts of mathematics to be brought together to reach a solution.

A Level Mathematics Working Group - Report on Mathematical Problem Solving, Modelling and the Use of Large Data Sets in Statistics in AS/A Level Mathematics and Further Mathematics. Ofqual 2015

The report contains a footnote that indicates that not all of the attributes would be required for a task to be considered to be problem solving. It also points out that a task having one or more of these attributes does not always make it a problem solving task.

Example 19 - AS/A Level

Solution to example 19

For the *OCR A Level Mathematics B (MEI)* specification, questions with the introductory sentence "In this question you must show detailed reasoning." often involve students having to use problem solving skills as well as the detailed reasoning requested.

Example 20

Solution to example 20

GCSE

The Mathematics GCSE Subject Content and Assessment Objectives (DfE, 2013) states that specifications in mathematics should enable students to acquire, select and apply mathematical techniques to solve problems. The problem solving assessment objective (AO3) should comprise of 30% of the higher tier marks available and 25% of the foundation tier marks. There is a strong emphasis on translating problems into a series of mathematical processes, making and using connections between different aspects of GCSE mathematics, interpreting results and evaluating results. Often, questions providing marks for AO3 are linked to AO2: reason, interpret and communicate mathematically.

Foundation Tier

Problems at Foundation Tier are no less challenging to students even though they require more basic mathematical skills. Here are some examples of the sort of problems that students may encounter at Foundation Tier:

Example 21 – GCSE Foundation Tier

Solution to example 21

Example 22 – GCSE Foundation Tier

Solution to example 22

Example 23 – GCSE Foundation Tier

Solution to example 23

Higher Tier

Problems at Higher Tier range from those that use skills from Foundation Tier and could be used at both levels to those that employ a number of Higher Tier mathematical skills. Here are some examples of problems that students could encounter at Higher Tier:

Example 24 – GCSE Foundation/Higher Tier

Solution to example 24

Example 25 – GCSE Higher Tier

Solution to example 25

These examples give some idea of the variety of problems that students could encounter. The main things that they have in common are:

- There is little or no scaffolding in the problem. Students have to find their own path.
- There is often the need to make and use connections between different parts of the mathematics that students have been taught.
- There are often multiple paths to the solution.

It is important therefore that students are given the opportunity throughout their course to attempt and solve a variety of problems so they are best able to cope with whatever problems turn up in the final examination.

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