Mathematical Problem Solving
AS/A Level example

Example 7

A class have been studying the trigonometric addition and double angle formulae.

The class are separated into pairs (the teacher had one set of 3 problems ready in case there was an odd number of students).

One pair received these two problems:

Student A

You have been taught the following identities:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}, \qquad A + B \neq k + \dfrac{\pi}{2}$

You should know the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for $\theta = 0°, 30°, 45°$ and $60°$ and the sine and cosine of $90°$.

Problem

Part (i)

Find an expression for $\sin 75°$ in the form $\frac{\sqrt{a}+\sqrt{b}}{c}$, where a, b and c are positive integers.

Part (ii)

Find the value of $\sin 75° - \cos 75°$.

Give your answer in surd form.

Student B

You have been taught the following identities:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}, \qquad A + B \neq k + \dfrac{\pi}{2}$

$\sin 2A = 2\sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\sin^2 A + \cos^2 A = 1$

You should know the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for $\theta = 0°, 30°, 45°$ and $60°$ and the sine and cosine of $90°$.

Problem

Part (i)

Find an expression for $\cos 3A$ in the form

$a\cos^n A + b\cos A$ where a, b and n are integers

Part (ii)

Given that $\cos 3x = 1$, and without finding the value of x, show that there are two possible values of $\cos x$ and find these values.

The students were allowed around 7 minutes to answer only parts (i) of their problems. This was based on the teacher's observations about progress through the given problems for the whole class.

The students then each spent 5 minutes explaining to their partner what they had done followed by 2 minutes answering any questions about their explanation.

The students then swapped and attempted to complete parts (ii) of each other's problems.

After another 7 minutes, the students were asked to finish off any calculations and then at 8 minutes to stop working.

The students then each spent another 5 minutes explaining to their partner what they had done. This was again followed by 2 minutes for each of the pair to ask any questions.

The activity took around 45 minutes to complete.

The teacher then picked up some of the incomplete solutions and modelled their completion.

Commentary

The two students given the above problems gave the following paraphrased explanations and responses to questions. One student was more able than the other and the problems were selected accordingly. All of the calculations have been tidied for clarity.

Student A

Part (i)

Working

$\sin 75 = \sin(45 + 30)$

$= \sin 45 \cos 30 + \cos 45 \sin 30$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$

Student B

Part (i)

Working

$\cos 3A = \cos(2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (\cos^2 A - \sin^2 A) \cos A - 2 \sin A \cos A \cdot \sin A$

$= \cos^3 A - \sin^2 A \cos A - 2 \sin^2 A \cos A$

$= \cos^3 A - 3 \sin^2 A \cos A$

$= \cos^3 A - 3(1 - \cos^2 A) \cos A$

$= \cos^3 A - 3 \cos A + 3 \cos^2 A$

$= 4 \cos^3 A - 3 \cos A$

Student A	**Student B**
Explanation I knew I had to use one of these (indicating the initial list of identities) with the standard values we know and I worked out that I could do it with $45 + 30$ so I put those in to $\sin(A + B)$. I could remember $\sin 30$ and $\sin 45$ and $\cos 45$ but not $\cos 30$ so I asked you. I put those in and got this (indicates the line of calculation). They're the same type of fractions so I could just put them together. All I had to do then was multiply top and bottom by $\sqrt{2}$ as there was no $\sqrt{2}$ at the bottom of the way they wanted the answer.	**Explanation** I knew I had to use the identities too and guessed it would be more than one of them. The only way I could see to get $\cos 3A$ was to use $\cos(2A + A)$ so that's what I did here (indicates the line of calculation). Looking at the final form the answer had to be in I could see that I didn't want to have either $\sin 2A$ or $\cos 2A$ in it. I just swapped the $\sin 2A$ for $2 \sin A \cos A$ and the $\cos 2A$ for $\cos^2 A - \sin^2 A$ which gave this (indicates the line of calculation). I multiplied out the brackets and simplified by adding these two terms together (indicates the $\sin^2 A \cos A$ terms). The last bit was changing the $\sin^2 A$ term to $1 - \cos^2 A$ which I did here. After multiplying out the brackets and simplifying the terms I finally got the answer in the form they wanted.
Questions B: How did you know to use $\sin(45 + 30)$ rather than anything else? A: It was the only way I could think of to make 75 that would work with any of the identities. It couldn't have been done with the double angle ones as that would need $37\frac{1}{2}$. Note: B was satisfied by the explanation given by A and so the remainder of the two minutes was spent tidying up A's presentation of the solution.	**Questions** A: How did you know to use $\cos(2A + A)$? B: A bit like you did when you used $45 + 30$. There's no other way to get $3A$ that would fit with any of the identities we were given. A: How did you know how to change the $\sin 2A$ and $\cos 2A$ terms into those terms? (indicates $2 \sin A \cos A$ and $\cos^2 A - \sin^2 A$)

Student A	**Student B**
	B: I didn't really know it would work. I looked at the way I had to write the answer and could see the $\cos^3 A$ so I could see that using $\cos^2 A - \sin^2 A$ would bring it in. I wasn't sure of the other bit until I saw what I got and then I realised I could use $\sin^2 A + \cos^2 A = 1$ which we seem to do a lot.

Problems swapped (Student A)

Part (ii)

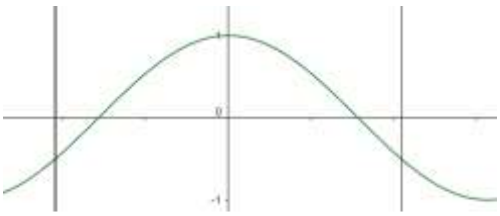
Working

$\cos 3x = 1$

$3x = 0, 360, -360$

$x = 0, 120, -120$

$\cos 0 = 1$



$\cos 120 = \cos -120 = -\cos 60 = -\frac{1}{2}$

$4\cos^3 x - 3\cos x = 1$

$4\cos^3 x - 3\cos x - 1 = 0$

Problems swapped (Student B)

Part (ii)

Working

$\cos 75 = \cos(45 + 30)$

$= \cos 45 \cos 30 - \sin 45 \sin 30$

$= \frac{1}{\sqrt{2}} \cdot \frac{1}{2} - \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

$\sin 75 - \cos 75 = \frac{\sqrt{6} + \sqrt{2} - (\sqrt{6} - \sqrt{2})}{4} = \frac{2\sqrt{2}}{4}$

$= \frac{\sqrt{2}}{2}$

Student A

Explanation

I couldn't see how to do it without finding x so I just did that otherwise I wouldn't have got an answer.

I sketched $\cos x$ so I could see the values where it's 1. Then I divided them by 3 which gave -120, 0 and 120.

I looked at my graph to see where -120 and 120 were and realised they must be something like 60.

Since $\cos 60 = \frac{1}{2}$, $\cos 120$ and $\cos -120$ are both $-\frac{1}{2}$.

So the two values of $\cos x$ are 1 (from 0) and $-\frac{1}{2}$ from the 120s.

I then thought that I should have used what you did from part (i) and wrote it in here (indicates working) but I couldn't work out what to do next.

Questions

This turned in to B helping A finish the problem using the intended identity.

B: You've got as far as $4\cos^3 x - 3\cos x - 1 = 0$. What stopped you here?

A: I realise that it is like one of those ones that gives a quadratic equation but we've got a cube here.

B: We did this when we did the factor theorem. I'll write this as $4c^3 - 3c - 1 = 0$.

Look at the numbers, if we use $c = 1$ it'll definitely work so $c - 1$ goes into it.

A: Is this where we use that long division?

B: Yes, but I don't do that, I do this:

Student B

Explanation

I just though the easiest thing to do would be to find $\cos 75$ just like you found $\sin 75$ so I used $\cos(A + B)$ and subbed the numbers in.

It gave me almost the same thing as you got for $\sin 75$.

After that I just took my $\cos 75$ from your $\sin 75$.

I got it wrong first time as I forgot to change the sign on the $-\sqrt{2}$ but it didn't feel right so I did it again and I think this is the right answer (indicates the correct solution).

It's the same as $\sin 45$ and $\cos 45$ but I can't see why that is.

Questions

A: I can see what you've done but can you talk me through the final subtraction bit as I can't see why you added the $\sqrt{2}$s together.

B: It's best if you put brackets around the top of each fraction. You see that they've got the same denominator so you can just combine them. When you subtract the top of that fraction (indicates the right hand side one), the whole of the top is being subtracted so you have minus root 6 but minus, minus root 2 which gives you the plus root 2.

Student A	Student B

$(c-1)(4c^2+kc+1)=4c^3-3c-1$

A: Is that the division?

B: No, I'm trying to write the cubic in a different way. If we times the $c-1$ by a quadratic, we can get a cubic. The first term has to be $4c^2$ to give $4c^3$. The last term has to be $+1$ to give the -1 at the end. The only term I don't know is the middle one so I've just said it's plus something times c.

You can get that something by equating coefficients of c^2. For these two brackets, these multiplications (draws lines) give c^2 terms. So there are $(-4+k)$ c^2 terms on the left. There are no c^2 terms and this is supposed to be the same thing so $-4+k$ must be 0 so $k=4$.

B: That means $4c^3-3c-1$ can be written as $(c-1)(4c^2+4c+1)$. All we've got to do now is solve $(c-1)(4c^2+4c+1)=0$

A: The second bracket factorises.

It's $(2c+1)(2c+1)$

B: So we get $c=1$ from the first bracket and both of the others give $c=-\frac{1}{2}$ which you got.

It is worth noting that, although the students did not follow the exact instructions for the activity, the dialogue shows that they both gained some useful experience from it. Student A, even though they could not complete part (ii) of the harder problem, benefited from having to explain what they had done, as well as from being shown how to combine several methods to solve it. Student B benefited by having to organise their thoughts and consider how best to explain something to another student.