Mathematical Problem Solving
GCSE example

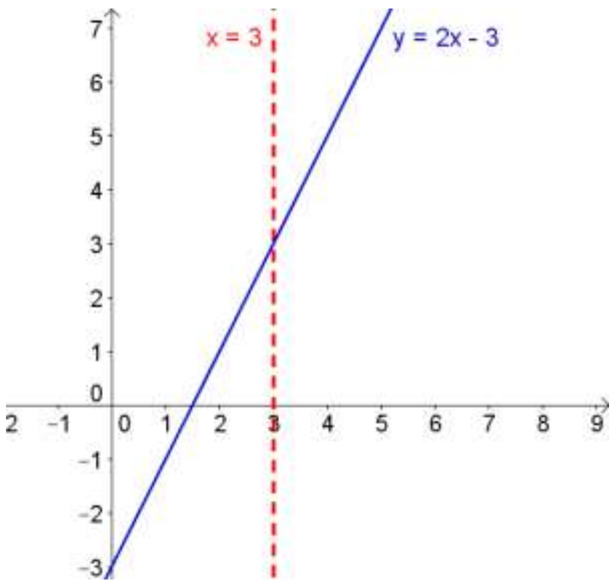
Example 8

A class have been studying gradient and the equation of a straight line.

The class are separated into pairs.

One pair received these two problems:

Student A



Problem

Part (i)

Reflect the line $y = 2x - 3$ in the line $x = 3$.

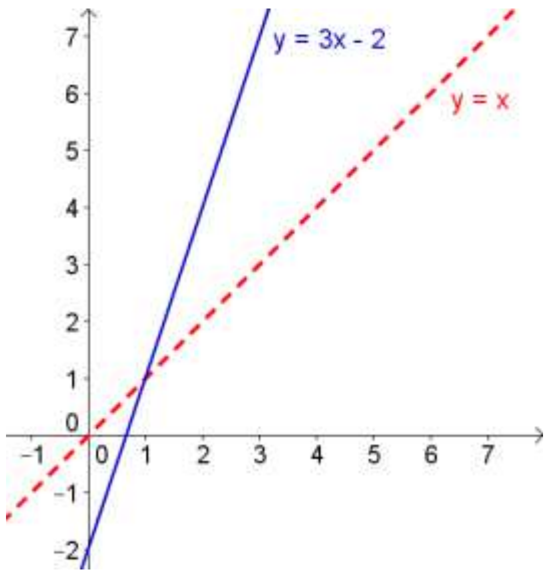
What is the equation of the new line?

Part (ii)

What equation would you get if you reflect the line $y = 2x - 3$ in $x = 4$?

Is there a rule for reflecting $y = 2x - 3$ in any line of the form $x = k$?

Student B



Problem

Part (i)

Reflect the line $y = 3x - 2$ in the line $y = x$.

What is the equation of the new line?

Part (ii)

What equation would you get if you reflect the line $y = 5 - 2x$ in $y = x$?

Is there a rule for reflecting a line of the form $y = mx + c$ in the line $y = x$?

The students were allowed around 7 minutes to answer part (i) of the problem. This was based on the teacher's observations about progress through the given problems for the whole class.

The students then each spent 5 minutes explaining to their partner what they had done followed by 2 minutes answering any questions about their explanation.

The students then swapped problems and attempted to complete each other's problems.

After another 7 minutes, the students were asked to finish off any calculations and then at 8 minutes to stop working.

The students then each spent another 5 minutes explaining to their partner what they had done. This was again followed by 2 minutes for each of the pair to ask any questions.

The activity took around 45 minutes to complete.

The teacher then picked up some of the incomplete solutions and modelled their completion.

Commentary

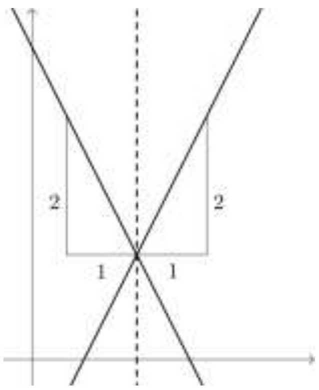
The two students given the above problems gave the following paraphrased explanations and responses to questions. The students were of similar high ability. All of the calculations have been tidied for clarity.

Student A

Part (i)

Working

The student sketched an equivalent of this diagram



Gradient of new line $= -2$

$x = 3$

Student B

Part (i)

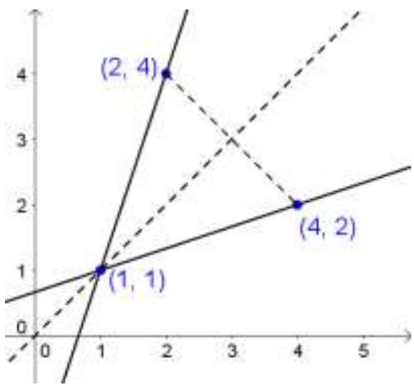
Working

Cross at $(1,1)$

$y = 3x - 2$

$x = 2,\ y = 3 \times 2 - 2 = 4$

Drawn on original diagram



Student A

$y = 2 \times 3 - 3 = 3$

It goes through $(3,3)$

$y = -2x + c$

$3 = -2 \times 3 + c$

$3 = -6 + c$

$c = 9$

The line is

$y = -2x + 9$

Student B

Goes through $(4,2)$

New gradient $= \frac{2-1}{4-1} = \frac{1}{3}$

$y = \frac{1}{3}x + c$

It goes through $(1,1)$

$1 = \frac{1}{3} + c$

$c = \frac{2}{3}$

The line is

$y = \frac{1}{3}x + \frac{2}{3}$

Explanation (Student A)

We always find these out from the gradient so I thought I needed the gradient of the new line.

I knew the old line had a gradient of 2 as it was $y = 2x - 3$.

A gradient of 2 is one across and two up so I did a sketch of it as a triangle. From that I could see that the reflected line had a gradient of -2 as it was the same but downhill.

The other thing we've always used is a point. From my sketch I could see that the two lines crossed on the mirror so $x = 3$. I just used the first line to work out the y that went with that. That gave 3 too.

To get the equation of the line I just stuck the -2 gradient in to $y = mx + c$ and then used $x = 3$ and $y = 3$ to get c.

Explanation (Student B)

I could see from the diagram that the line and the mirror passed through $(1,1)$ so I knew the reflection would pass through it too.

I couldn't think how to get the gradient without a point so I found a point on the $3x - 2$ line by using $x = 2$ which gave $y = 4$.

I used the picture I was given to reflect that point and I got $(4,2)$. I think it always swaps coordinates when you reflect in $y = x$.

I used $(1,1)$ and $(4,2)$ to work out the new gradient using that formula we were given.

I knew the line went through $(1,1)$ so I subbed that in to $y = \frac{1}{3}x + c$ to get c.

Student A	Student B
Questions B: How did you know that the gradient was the same? A: If I draw this triangle for the gradient on the first line and reflect it in $y = x$ it will be the same triangle but back to front so it's downhill rather than uphill.	**Questions** A: Were you sure the diagram was accurate and that they did cross at $(1,1)$? B: It looked like it, I didn't check. $(1,1)$ is definitely on $y = x$ so I thought it was probably OK for the other one too. The students checked that $(1,1)$ was on $y = 3x - 2$ here. A: As reflecting in $y = x$ swaps the x and y numbers, do you reckon your equation is the same as $x = 3y - 2$? B: I don't know. Can we check it At this stage they ran out of time for the Q&A part so didn't actually check this idea.
Problems swapped Part (ii) Working $x = 5 - 2y$ $2y = 5 - x$ $2y = -x + 5$ $y = -\frac{1}{2}x + \frac{5}{2}$ $y = x$, $y = 5 - 2x$ $x = 5 - 2x$ $3x = 5$ so $x = \frac{5}{3}$ and $y = \frac{5}{3}$ Goes through $\left(\frac{5}{3}, \frac{5}{3}\right)$ Gradient $-2 \rightarrow -\frac{1}{2}$	**Problems swapped** Part (ii) Working $x = 4$, $y = 2 \times 4 - 3 = 5$ Point $(4,5)$ Gradient $= -2$ $y = -2x + c$ $5 = -2 \times 4 + c$ $5 = -8 + c$ $c = 13$ $y = -2x + 13$ $x = k$, $y = 2k - 3$ Gradient $= -2$

Student A	**Student B**
$y = -\frac{1}{2}x + c$ $\frac{5}{3} = -\frac{5}{6} + c$ $c = \frac{5}{3} + \frac{5}{6} = \frac{10+5}{6} = \frac{15}{6} = \frac{5}{2}$ $y = -\frac{1}{2}x + \frac{5}{2}$ a) Swap the x and the y and then rearrange it to $y = mx + c$	$y = -2x + c$ $2k - 3 = -2k + c$ $c = 4k - 3$ $y = -2x + 4k - 3$ The new line will be $y = -2x + 4k - 3$
Explanation I'd guessed that I could just swap x and y so I did that and rearranged it to $y = mx + c$ I thought I'd better check that it worked by actually working out the equation so I found out where the line crossed the mirror here (indicates appropriate line of working). I knew the gradient is going to be 1 over the old gradient like it was before so I used that and my point to find the equation of the line and it did come out the same That meant that I could just write out what happens for part b).	**Explanation** I needed the crossing point so I put in $x = 4$ to the line and got $y = 5$. I could see from your triangle reflection picture that the gradient would always be -2 if you reflect in a vertical. I used the two bits of information to get the equation of the line. For part b) I realised that I needed the point where the line crossed $x = k$ so I could find c as I knew the gradient would be -2. We've had to do a few things like this where we don't use numbers so I just went with it and found out that when $x = k$, $y = 2k - 3$ which seemed obvious when I did it. I then subbed this into $y = -2x + c$ to get c.

Student A

Questions

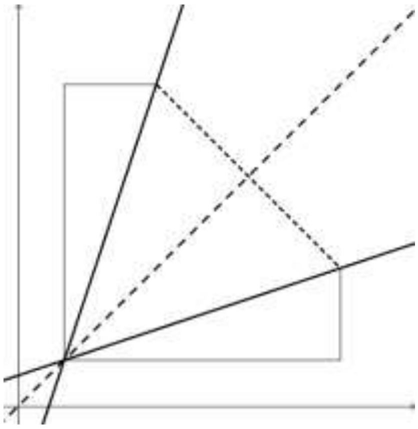
B: I don't get how you knew the gradient was going to be 1 over the old gradient.

A: It's what happened when you did the first part so I guessed it would happen again.

B: Are you sure that always happens?

A: I think it does.

A then drew this sketch



A: Yes, the x and y swap so we'll get the reciprocal which is the same as 1 over it.

Student B

Questions

A: What did you do to get the value of c again?

B: I knew I needed a coordinate where the lines crossed to get c so I use the x value of the mirror line to find the y value where it crossed $y = 2x - 3$. As the mirror was $x = k$, it came out as $2k - 3$. All I did was sub that x and y into my new equation to get c as something with k in it.

A: Oh, I see.

At this point student A seemed satisfied with the explanation given by student B.