Solution to example 18

Find the values of a such that the turning point (i.e. the vertex) of the parabola $y = x^2 - 2ax + 1$ is closest to the origin.

What is the locus of the turning point as a varies?

The equation can be rewritten as $y = (x - a)^2 + 1 - a^2$ which gives the vertex of the parabola as $(a, 1 - a^2)$

The distance of this point from the origin is $\sqrt{a^2 + (1 - a^2)^2} = \sqrt{a^2 + 1 - 2a^2 + a^4} = \sqrt{a^4 - a^2 + 1}$

The distance will be at a minimum when $a^4 - a^2 + 1$ is at a minimum

By completing the square $\left(a^2 - \frac{1}{2}\right)^2 + 1 - \frac{1}{4}$, we see that the minimum value will occur when

$a^2 = \frac{1}{2}$ i.e. $a = \pm\frac{1}{\sqrt{2}}$

From $(a, 1 - a^2)$, $x = a$ and $y = 1 - a^2$ so the locus is $y = 1 - x^2$.