The irrationality of $\sqrt{2}$

Teacher Notes

Hand out the 22 cards below to each pair of students and display the final page.

Points to note about this task:

- The shorter proof relies on knowledge of prime factorisation; what pre-tasks would prepare students for this?
- The initial focus is on the structure of the proof – which cards go at the beginning and the end? – so students are learning about proof by contradiction before worrying about details.
- Students have to think about the hcf=1 requirement in the standard proof and also why it is not needed in the other proof.
- Working in pairs on this means that students are 'comprehending and critiquing mathematical arguments', both those on the cards and what their partners are saying.
- Students get to compare proofs, which is an important follow up activity; this includes thinking how to extend the proofs to the irrationality of $\sqrt{3}$ (and possibly $\sqrt{10}$ and $\sqrt{12}$, all subtly different) and thinking about why they fail to prove that $\sqrt{4}$ is irrational.

A Suppose, for a contradiction, that $\sqrt{2}$ is rational	**A** Suppose, for a contradiction, that $\sqrt{2}$ is rational
M This is a contradiction, so our original assumption that $\sqrt{2}$ is rational must be wrong	**M** This is a contradiction, so our original assumption that $\sqrt{2}$ is rational must be wrong
G That is, we can write $\sqrt{2} = \frac{m}{n}$ where m and n are integers and where $n \neq 0$	**G** That is, we can write $\sqrt{2} = \frac{m}{n}$ where m and n are integers and where $n \neq 0$
J Squaring, we have $2 = \frac{m^2}{n^2}$	**J** Squaring, we have $2 = \frac{m^2}{n^2}$
D By cancelling any common factors we can assume that the h.c.f. of m and n is 1	**K** But prime factorisations are unique, so 2 should appear to the same power in both $2n^2$ and m^2

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E Multiply across to get $2n^2 = m^2$	**E** Multiply across to get $2n^2 = m^2$
I So $\sqrt{2}$ is irrational	**I** So $\sqrt{2}$ is irrational
B So $n^2 = 2k^2$ And n is even	**P** In the prime factorisation of m^2 and n^2, 2 occurs to an even power
F So $2n^2 = 4k^2$	**L** m and n have a common factor of 2
C This shows that m^2 is even	**O** So m is even
H In the prime factorisation of $2n^2$, 2 occurs to an even power	**N** Let $m = 2k$ where k is an integer

Proving the irrationality of $\sqrt{2}$

There are 22 cards. They contain, muddled up, two different proofs that $\sqrt{2}$ is irrational. One of the cards has a mistake on it.

Sort the cards into the right order so that you have two proofs.

Compare the proofs. Which do you prefer, and why? Do you want to improve either of the proofs; e.g. by adding extra lines or explanations?

Adapt one of the proofs to show that $\sqrt{3}$ is irrational.

Where does each proof break down if you try to show that $\sqrt{4}$ is irrational?

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