Sequences and Series

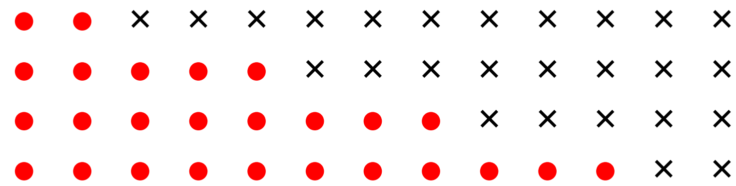
D2	Work with sequences including those given by a formula for the *n*th term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}\left(x_n\right)$; increasing sequences; decreasing sequences; periodic sequences
D3	Understand and use sigma notation for sums of series
D4	Understand and work with arithmetic sequences and series, including the formulae for *n*th term and the sum to *n* terms
D5	Understand and work with geometric sequences and series including the formulae for the *n*th term and the sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $\|r\| < 1$; modulus notation
D6	Use sequences and series in modelling

Commentary

This topic is just about number sequences so take the opportunity to ask questions based on a spreadsheet, as in the technology resource below. The mathematical notation and terminology is there to help. All students should be able to make some progress with the sample resource below; this could be used to analyse what support they need rather than spending time teaching these techniques.

Students are often presented with formulae before they need, or are ready for, them. Consider the mathematical thinking in the following problem and how the use of the geometric representation in advance of the algebraic representation prepares students for the formula for the sum of an arithmetic series.

How does this image show that $2+5+8+11=\frac{4\times 13}{2}$?
Using this idea work out $8+13+18+...+93+98$.



Look for opportunities to use the geometrical properties of square and triangular numbers. Why is any triangular number multiplied by 8 one less than a square?

Working with $0.\dot{9}$ as an infinite geometric series provides an alternative to the usual GCSE approach of subtracting $x=0.\dot{9}$ from $10x=9.\dot{9}$ to appreciate why $0.\dot{9}=1$. Explore other number bases; for example, what is the binary number $0.\dot{0}\dot{1}$?

Challenge perceptions of infinity and limits by contrasting the converging geometric series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ and the diverging harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, and, to reinforce connections, compare these series with the definite integrals $\int_1^n 0.5^x \mathrm{d}x$ and $\int_1^n \frac{1}{x}\mathrm{d}x$ as n increases.

Many common sequences, such as the square numbers and the Fibonacci sequence, are neither arithmetic nor geometric; these are studied in Further Maths.

Sample MEI resource

'Thinking about sequences' (which can be found at https://my.integralmaths.org/integral/sow-resources.php) encourages students to convert between numeric and algebraic representations; after seeing the example in the first column and working in pairs they should be able to fill in the other cells. You might want to display the terminology that will be needed in the fifth row.

Thinking about sequences

	$a_n = 3^n$	$b_n = 2n+3$	$c_n =$	$d_n =$	$u_n =$
First few terms	$3, 9, 27, 81, \ldots$		$9, 5, 1, -3, \ldots$		
Inductive definition	$a_{n+1} = 3 \times a_n$, $a_1 = 3$			$d_{n+1} = \frac{1}{2} d_n$, $d_1 = 4$	$u_{n+1} = -3 \times u_n$, $u_3 = 18$
Specific term	$a_{20} = 3^{20}$	$b_{100} =$	$c_{50} =$	$d_{10} =$	$u_{21} =$
Associated terminology	Geometric Diverging				
Sum of terms	$\sum_{n=1}^{5} a_n = 363$	$\sum_{n=1}^{10} b_n =$	$\sum_{n=1}^{12} c_n =$	$\sum_{n=4}^{8} d_n =$	$\sum_{n=1}^{7} u_n - \sum_{n=1}^{6} u_n =$

On completion, ask students to make up two columns of their own, one easy and one difficult, and to describe what it is about a question that makes it difficult.

Effective use of technology

Make use of a spread sheet to emphasise that this topic is simply about number sequences. For example:

	A	B
1	**sequence**	**series**
2	3	3
3	7	10
4	11	21
5	15	36
6	19	55
7	23	78
8	27	105
9	31	136

If the table is extended,

- What's the first four digit number in the sequence?
- If the sequence started with 31 and the common difference was -4 what would you see in the series column?
- There are no square numbers so far in the sequence. Either find when the first one appears or prove there aren't any.

## Sequences and Series	**Time allocation:**

Pre-requisites

- GCSE: Knowledge of the nth term of a sequence, although in arithmetic sequences at GCSE this is unlikely to involve the use of $n-1$ additions of the common difference.
- Logarithms and exponentials: logs can be used to solve some equations related to geometric sequences.
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Links with other topics

- Logarithms and exponentials: Compare the geometric sequence $u_n = 2^n$ and the exponential function $\mathrm{f}(x) = 2^x$.
- Algebra: The binomial theorem starts with $(1-x)^{-1}$, the geometric series starts with $1+x+x^2+...$ and both arrive at the same result.
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Questions and prompts for mathematical thinking

- Explain connections between infinite geometric series and recurring decimals.
- Explain connections between the area of a trapezium and summing arithmetic series.
- Give me an example of an infinite geometric series with sum 4…and another…and another.
- Prove that, for every triangular number T, $8T+1$ is a square number.
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Opportunities for proof

- Prove that the infinite arithmetic sequence 3,7,11,15,19,... contains no square numbers.
- Prove the formulae for the sum of arithmetic and geometric series
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Common errors

- Using n rather than $n-1$ in the formulae for n^{th} term in both arithmetic and geometric sequences.
- Mixing up the formulae for n^{th} term and the sum of n terms.
- Weak algebra. For example, inability to simplify $\frac{32(1.25^n - 1)}{1.25-1}$ or to solve $1-r^3 = 0.488$.
- Errors using logarithms in summing geometric series; e.g. in solving $1-\left(\frac{7}{8}\right)^n > 0.9$
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