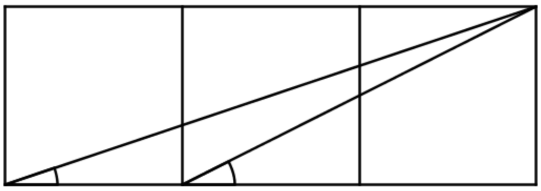
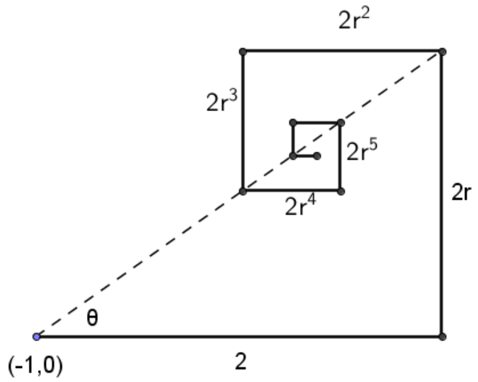
Three challenge questions

1. The diagram shows three congruent squares. Find the sum of the two angles marked.

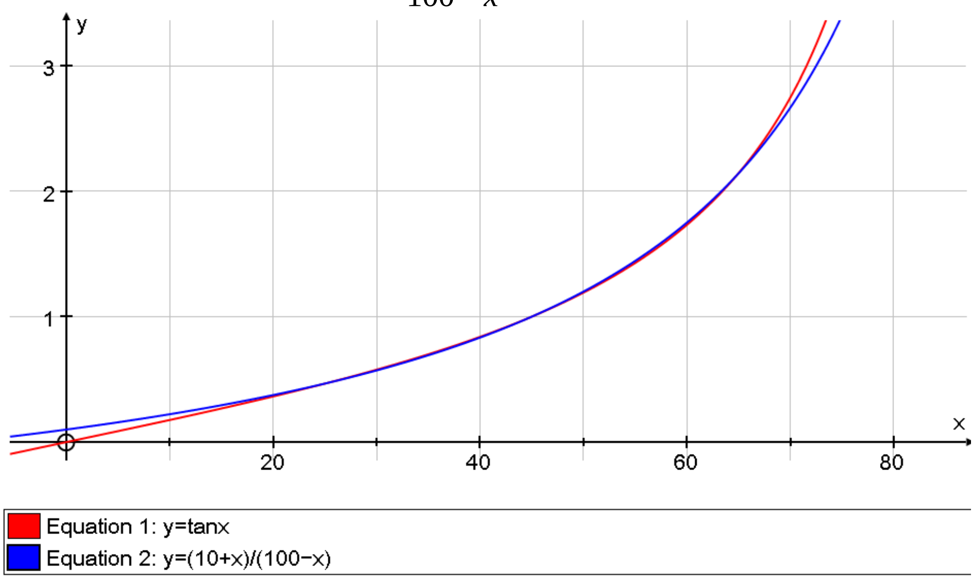


2. The spiral below starts at the point $(-1,0)$ and the perpendicular edges are drawn in an anticlockwise spiral with a common ratio r as shown.



Due to similarity, after an even number of steps the leading point will be on the diagonal line shown. If this diagonal makes an angle θ with the first edge as shown, find the coordinates of the point on which the spiral is converging.

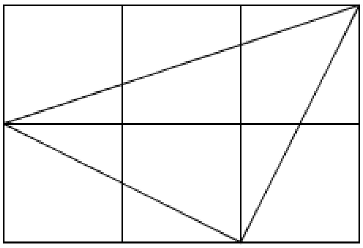
3. Investigate the approximation $\tan x^\circ \approx \dfrac{10+x}{100-x}$.



Innovators in Mathematics Education
SC 290716
Version 1.0

Hints and solutions

1. This can be done using the compound angle formulae for $\tan(\alpha+\beta)$ but it is more satisfying to prove this using the diagram below. How?

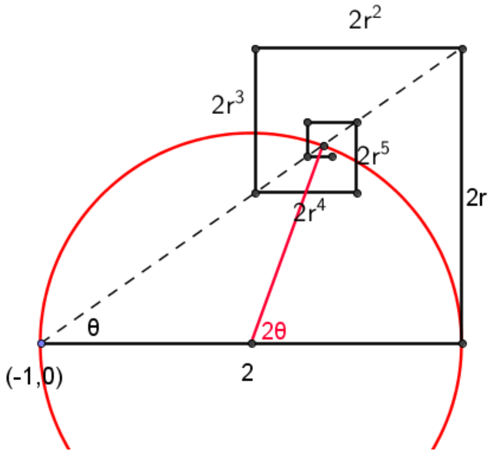


2. The spiral homes in on the point (X,Y) where (using $r=\tan\theta$)

$$X=(-1)+2-2r^2+2r^4-...=-1+\frac{2}{1+r^2}=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}=\cos 2\theta$$

and $Y=2r-2r^3+2r^5-2r^7+...=\dfrac{2r}{1+r^2}=\dfrac{2\tan\theta}{1+\tan^2\theta}=\dfrac{2\sin\theta\cos\theta}{\cos^2\theta+\sin^2\theta}=\sin 2\theta$

Therefore the spiral converges on the point $(\cos 2\theta,\sin 2\theta)$. Think how this links in with the circle theorems.



3. For small values of x, in radians, $\tan x\approx x$ and so $\tan x^\circ\approx\dfrac{\pi x^\circ}{180^\circ}$ for sufficiently small x.

$$\tan x^\circ=\tan\left(\left(x^\circ-45^\circ\right)+45^\circ\right)=\frac{\tan\left(x^\circ-45^\circ\right)+\tan 45^\circ}{1-\tan\left(x^\circ-45^\circ\right)\tan 45^\circ}\approx\frac{\frac{\pi}{180^\circ}\left(x^\circ-45^\circ\right)+1}{1-\frac{\pi}{180^\circ}\left(x^\circ-45^\circ\right)}$$

$$=\frac{\pi(x-45)+180}{180-\pi(x-45)}=\frac{x+\left(\frac{180}{\pi}-45\right)}{\left(\frac{180}{\pi}+45\right)-x}\approx\frac{x+10}{100-x}$$