Integration (A level)

H2	Integrate e^{kx}, $\frac{1}{x}$, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples
H3	Use a definite integral to find the area between two curves
H4	Understand and use integration as the limit of a sum
H5	Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively (Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae)
H6	Integrate using partial fractions that are linear in the denominator

Commentary

It is helpful to introduce the two new rules for integration as the reverse of two of the rules met in the differentiation unit: the reverse of the chain rule is integration by substitution and the reverse of the product rule is integration by parts.

Students now know quite a lot of techniques for integration. Which of these can they do and which need more advanced techniques?

$$\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x, \quad \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x, \quad \int \frac{1}{\sqrt{x^2-1}}\mathrm{d}x, \quad \int \frac{x}{\sqrt{x^2-1}}\mathrm{d}x$$

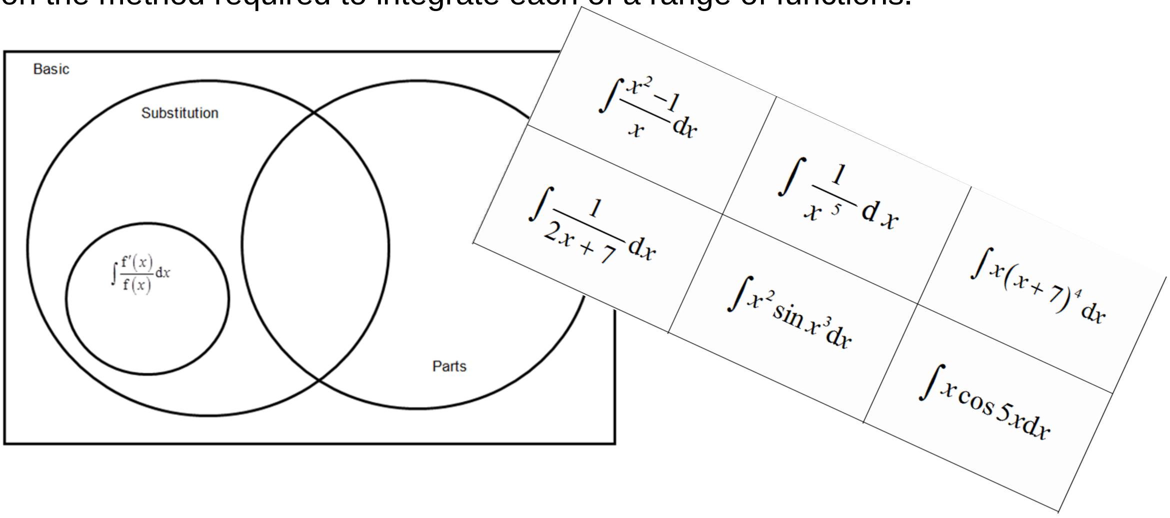
$$\int \frac{1}{1-x^2}\mathrm{d}x, \quad \int \frac{x}{1-x^2}\mathrm{d}x, \quad \int \frac{1}{1+x^2}\mathrm{d}x, \quad \int \frac{x}{\sqrt{1+x^2}}\mathrm{d}x$$

It is important not to perform integration in isolation. For example, when asked to evaluate $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin x + \cos 2x \,\mathrm{d}x$ students might first attempt to sketch the curve, thinking about symmetry, in order to appreciate that they could instead simply evaluate $2\int_0^{\frac{\pi}{3}} \cos 2x \,\mathrm{d}x$. Similarly, what does $y = \frac{2x}{x^2-1}$ look like (and can students find two ways of integrating this function?)

When using substitution to evaluate definite integrals, it is interesting to look at the shape of the region under the original curve (y against x) and compare it with the region under the curve after substitution (y against u). Although the areas are the same, the shapes are often surprisingly very different. Similarly the shape of the region between two curves $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ can be very different from the shape of the region $\int_a^b \mathrm{f}(x) - \mathrm{g}(x)\,\mathrm{d}x$ and yet the areas are the same.

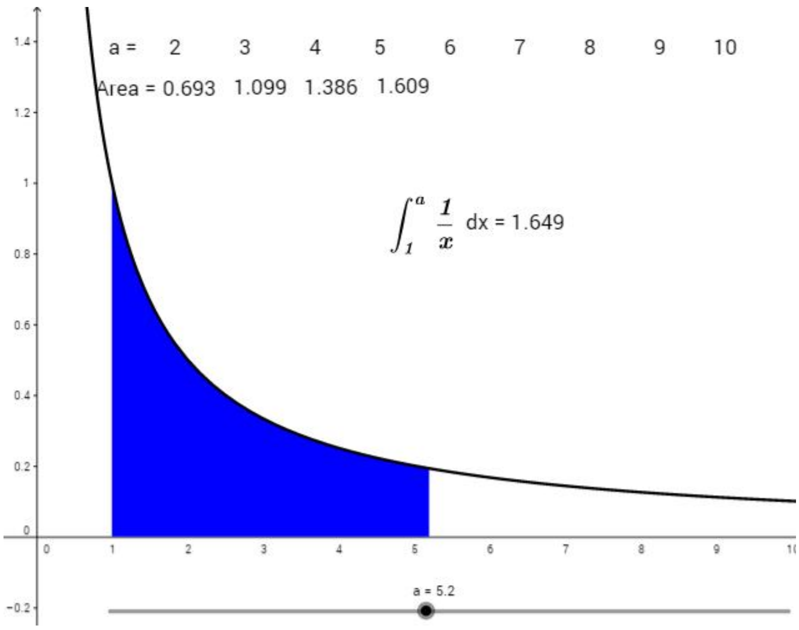
Sample MEI resource

‘Methods of Integration’ (which can be found at https://my.integralmaths.org/integral/sow-resources.php) is a group activity deciding on the method required to integrate each of a range of functions.



Effective use of technology

‘Area under 1/x and the natural logarithm function’ (which can be found at www.mei.org.uk/integrating-technology) provides a numerical way of illustrating how the area function $\text{Area}(a) = \int_1^a \frac{1}{x} \mathrm{d}x$ has logarithmic properties, such as $\text{Area}(a) + \text{Area}(b) = \text{Area}(ab)$.



Integration (A level)

Time allocation:

Pre-requisites

- Differentiation: knowing the derivatives of polynomial, trigonometric, exponential and logarithmic functions
- Integration (AS): definite and indefinite integrals
- Algebra: writing algebraic fractions in partial fraction form
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Links with other topics

- Trigonometric identities: Knowing $\sin^2 x + \cos^2 x = 1$ how can you evaluate $\int_0^{\pi} \sin^2 x \mathrm{d}x$ without using integration techniques?
- Functions: in terms of the symmetry possessed by the constituent functions, explain why $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin x + \cos 2x + x^3 \mathrm{d}x$ is the same as $2\int_0^{\frac{\pi}{3}} \cos 2x \mathrm{d}x$
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Questions and prompts for mathematical thinking

- Show me two ways of finding the indefinite integral $\int \sin x \cos x \mathrm{d}x$
- Give me an example of an integral which you could easily solve after making the substitution $u = \cos x$.
- Is substitution a useful technique for finding $\int_0^{\frac{\pi}{2}} x \sin x \mathrm{d}x$?
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Opportunities for proof

- Starting from the product rule for differentiation, prove $\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x$
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Common errors

- When using integration by substitution, forgetting to change the limits in definite integrals
- Inability to select the appropriate method (parts, substitution) for integrating a function
- The omission of $\mathrm{d}x$ in the student's original integrand and hence the failure to substitute for $\mathrm{d}u$.
- 'Chain rule integration'; e.g. $\int \sin 3x \, \mathrm{d}x = 3\cos 3x$ or $\frac{\cos 3x}{3}$ or $-\cos 3x$
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